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$$\begin{vmatrix} fx & , 1 , & \dfrac{\cos ax}{\cos az}, & \dfrac{1}{2^2}\dfrac{\cos 2ax}{\cos 2az}, & \ldots , & \dfrac{1}{n^2}\dfrac{\cos nax}{\cos naz}, & \dfrac{1}{(n+1)^2}\dfrac{\cos (n+1)ax}{\cos (n+1)az} \\ fy & , 1 , & \dfrac{\cos ay}{\cos az}, & \dfrac{1}{2^2}\dfrac{\cos 2ay}{\cos 2az}, & \ldots , & \dfrac{1}{n^2}\dfrac{\cos nay}{\cos naz}, & \dfrac{1}{(n+1)^2}\dfrac{\cos (n+1)ay}{\cos (n+1)az} \\ -\dfrac{1}{a^2}f''z & , 0 , & 1 , & 1 & , \ldots , & 1 & , \quad 1 \\ +\dfrac{1}{a^4}f^{\text{iv}}z & , 0 , & 1 , & 2^2 & , \ldots , & n^2 & , \quad (n+1)^2 \\ -\dfrac{1}{a^6}f^{\text{vi}}z & , 0 , & 1 , & 2^4 & , \ldots , & n^4 & , \quad (n+1)^4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \pm\dfrac{1}{a^{2n}}f^{2n}z & , 0 , & 1 , & 2^{2n-2} & , \ldots , & n^{2n-2} & , \quad (n+1)^{2n-2} \\ \Phi(u) & , 0 , & 0 , & 0 & , \ldots , & 0 & , \quad 1 \end{vmatrix} = 0 . \quad (129)$$

Expanding this with respect to the first and second rows, we obtain

$$fx - fy = \sum_1^n \frac{(-1)^{r+1}A_r}{r^2 \cos raz}(\cos rax - \cos ray) + R , \qquad (130)$$

in which A_r does not contain x.

This may be written

$$fx - fy = 2\sum_1^n \frac{(-1)^r A_r}{r^2 \cos raz} \sin \tfrac{1}{2}\, ra(x+y) \sin \tfrac{1}{2}\, ra(x-y) + R . \qquad (131)$$

In this put $x+h$ for x, and x for y, whence

$$f(x+h) = fx + 2\sum_1^n \frac{(-1)^r A_r}{r^2 \cos raz} \sin \tfrac{1}{2}\, rah \sin ra\,(x + \tfrac{1}{2}h) + R . \qquad (132)$$

In this put $x = 0$ and $h = x$, whence

$$fx = f0 + 2\sum_1^n \frac{(-1)^r A_r}{r^2 \cos raz} \sin^2 \tfrac{1}{2}\,(rax) + R . \qquad (133)$$

The value of A_r in these four formulæ is, when n is infinite,

$$A_r \doteq -2\sum_{p=1}^{p=\infty} \frac{1}{a^{2p}}\, {}^r\delta_p f^{2p}z .$$

In (130) put $z = y = l$, and $a = 2\pi/l$; then, since $\cos ral = 1$, we have

$$fx = A_0 + 2\sum_{r=1}^{r=n} \frac{(-1)^r}{r^2} A_r \cos \frac{2\pi r}{l} x + R , \qquad (134)$$

wherein, when n is infinite, we have for the constant term

$$A_0 = f(l) + 2 \sum_{r=1}^{r=\infty} \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}}{r^2} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(l)$$

$$= f(l) + \sum_{p=1}^{p=\infty} \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(l)$$

$$= \frac{1}{l} \int_{\frac{1}{2}l}^{\frac{3}{2}l} fx dx ,$$

in virtue of (124) and series (45).

The value of A_r is

$$A = \sum_{p=1}^{p=\infty} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p} z .$$

In like manner, if we had put $z = y = 0$, the constant term would have become

$$A'_0 = \sum_0^\infty \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(0) = \frac{1}{l} \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} fx dx ;$$

and the substitution $z = y = \frac{1}{2} l$ gives

$$A_0'' = \sum_0^\infty \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(0) = \frac{1}{l} \int_0^l fx dx ,$$

by series (47).

If, instead of $2\pi/l$, we put $a = \pi/l$; then, since $\cos r\pi = (-1)^r$, we have

$$fx = B_0 - 2 \sum_{r=1}^{r=n} \frac{1}{r^2} B_r \cos \frac{\pi r}{l} x + R , \tag{135}$$

wherein the constant term is, for $n = \infty$, $z = l$, $z = y = 0$, and $z = y = \frac{1}{2} l$, respectively,

$$B_0 = \sum_0^\infty \frac{1}{(2p+1)!} l^{2p} f^{2p}(l) = \frac{1}{2l} \int_0^{2l} fx dx ,$$

$$B_0' = \sum_0^\infty \frac{1}{(2p+1)!} l^{2p} f^{2p}(0) = \frac{1}{2l} \int_{-l}^{+l} fx dx ,$$

$$B_0'' = \sum_0^\infty \frac{1}{(2p+1)!} l^{2p} f^{2p}(\tfrac{1}{2} l) = \frac{1}{2l} \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} fx dx ;$$

the effect of the different substitutions being to change the limits between which the function is to be considered.

Expressions (134) and (135) are the equivalents of the familiar forms of the expansion in cosines.

If, in (134), we put $z = y = \frac{1}{2} l$, and multiply through by $2 \cos (2\pi r x/l)$, and integrate both sides from $x = 0$ to $x = l$, we get

$$\frac{1}{l} \int_0^l fx \cos \frac{2\pi r}{l} x \, dx = 2 \sum_{p=1}^{p=\infty} \frac{1}{r^2} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(\tfrac{1}{2} l),$$

the definite integral being the form of the coefficient as usually given.

It is to be observed that, if we make the substitution $z = y = \frac{1}{2} l$ and $a = 2\pi/l$ in (128), the signs in (134) become the same as the signs in (135), since $\cos (r\pi) = (-1)^r$; while the substitution $z = y = \frac{1}{2} l$ and $a = \pi/l$ requires us to throw out all columns containing odd values of r, since for these values $\cos \frac{1}{2} r\pi$ vanishes.

37. Consider the expansion of fx in terms of the functions $\sin rax$, or

$$\frac{1}{r} \sin rax,$$

since the columns will factor to this shape.

Remembering that in this case the composite can contain no even derivative rows in virtue of § 4, and proceeding exactly as in the last article, we deduce the formula

$$fx - fy = \sum_1^n \frac{(-1)^{r+1} A_r}{r \cos raz} (\sin rax - \sin ray) + R \tag{136}$$

$$= 2 \sum_1^n \frac{(-1)^{r+1} A_r}{r \cos raz} \cos \tfrac{1}{2} ra(x+y) \sin \tfrac{1}{2} ra(x-y) + R, \tag{137}$$

in which when n is infinite, we have

$$A_r = 2 \sum_{p=1}^{p=\infty} \frac{1}{a^{2p-1}} {}^r\mathfrak{F}_p . f^{2p-1}(z).$$

Put $a = 2\pi/l$ and $z = y = l$; then, since $\cos 2\pi r = 1$, we have

$$fx = f(l) + 2 \sum_{r=1}^{r=\infty} \frac{(-1)^{r+1}}{r} \sin \frac{2\pi r}{l} x \sum_{p=1}^{p=\infty} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(l), \tag{138}$$

assuming that n may be infinite.

If we put $z = y = \frac{1}{2} l$ instead of l, we have, since $\cos r\pi = (-1)^r$,

$$fx = f(l) + 2 \sum_{r=1}^{r=\infty} - \sin \frac{2\pi r}{l} x \sum_{p=1}^{p=\infty} \frac{1}{r} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(\tfrac{1}{2} l). \tag{139}$$

If fx be a function which vanishes with x, then $f(l) = 0$, and we may write (139)

$$f(-x) = \sum_{r=1}^{r=\infty} A_r \sin \frac{2\pi r}{l} x\,, \tag{140}$$

wherein

$$A_r = 2 \sum_{p=1}^{p=\infty} \frac{1}{r}\, {}^r\delta_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l)\,.$$

Multiply both sides of (140) by $2 \sin (2\pi r x/l)$ and integrate from $x = 0$ to $x = l$, whence

$$\frac{1}{l}\int_0^l fx \sin \frac{2\pi r}{l} x = -2 \sum_{p=1}^{p=\infty} \frac{1}{r}\, {}^r\delta_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l)\,,$$

the definite integral being the usual form of this coefficient.

We may derive Fourier's expansion of fx as a complex harmonic function, at once, by the addition of (140) and (134), but we deduce it directly from the composite on account of its importance and historical connection with this method of treatment.

38. Consider the expansion of fx in terms of the functions

$$\frac{1}{r^2} \cos rax \quad \text{and} \quad \frac{1}{r} \sin rax\,.$$

Making the substitutions in the composite, we have,

$$\left|\begin{array}{lllllll}
fx\,, & 1\,, & \cos ax\,, & \sin ax & \ldots\,, & \cos nax\,, & \sin nax\,, & (\mathrm{c},\ \mathrm{s}) \\
fy\,, & 1\,, & \cos ay\,, & \sin ay & \ldots\,, & \cos nay\,, & \sin nay\,, & (\mathrm{c},\ \mathrm{s}) \\
f'x_1\,, & 0\,, & -a \sin ax_1\,, & +a \cos ax_1 & \ldots\,, & -na \sin nax_1\,, & +na \cos nax_1\,, & (n+1)a\,(\mathrm{s},\ \mathrm{c}) \\
f''x_2\,, & 0\,, & -a^2 \cos ax_2\,, & -a^2 \sin ax_2 & \ldots\,, & -n^2a^2 \cos nax_2\,, & -n^2a^2 \sin nax_2\,, & (n+1)^2a^2\,(\mathrm{c},\ \mathrm{s}) \\
f'''x_3\,, & 0\,, & +a^3 \sin ax_3\,, & -a^3 \cos ax_3 & \ldots\,, & +n^3a^3 \sin nax_3\,, & -n^3a^3 \cos nax_3\,, & (n+1)^3a^3\,(\mathrm{s},\ \mathrm{c}) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
f^{2n-1}x_{2n-1}\,, & 0\,, & \mp a^{2n-1} \sin ax_{2n-1}\,, & \pm a^{2n-1} \cos ax_{2n-1}\,, & \ldots\,, & \mp (na)^{2n-1} \sin nax_{2n-1}\,, & \pm (na)^{2n-1} \cos nax_{2n-1}\,, & (\overline{n+1}\,.\,a)^{2n-1}(\mathrm{s},\ \mathrm{c}) \\
f^{2n}x_{2n}\,, & 0\,, & \mp a^{2n} \cos ax_{2n}\,, & \mp a^{2n} \sin ax_{2n} & \ldots\,, & \mp (na)^{2n} \cos nax_{2n}\,, & \mp (na)^{2n} \sin nax_{2n}\,, & (\overline{n+1}\,.\,a)^{2n}\,(\mathrm{c},\ \mathrm{s}) \\
\phi(u)\,, & 0\,, & 0\,, & 0 & \ldots\,, & 0\,, & 0\,, & 1
\end{array}\right| = 0. \tag{141}$$

The symbol (c, s) means that either cosine or sine may be used in the last column. In (141) put $a = 2\pi/l$, and

$$x_1 = x_2 = \ldots = x_{2n-1} = x_{2n} = \tfrac{1}{2}\, l$$

(noticing as we make this substitution what would result if, instead, we made the substitution $x_1 = \ldots = x_{2n} = 0$, or l).

We have throughout the body-determinant, $\sin r\pi = 0$ and $\cos r\pi = (-1)^r$.

Factor the a's from the rows, and the negative signs out of the body-determinant; continuing to write a for $2\pi/l$ for brevity. Expand (141) according to the method of § 30, and consider the determinant Fx. In Fx, run the first column and row to the middle, the sines to the left and cosines to the right, the odd derivative forms to the top and the even ones to the bottom; thus obtaining Fx in the following shape:

$$\begin{vmatrix}
1, & 1, & \ldots, & 1, & -\frac{1}{a}f'(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
1^2, & 2^2, & \ldots, & n^2, & +\frac{1}{a^3}f'''(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
1^4, & 2^4, & \ldots, & n^4, & -\frac{1}{a^5}f^{\mathrm{v}}(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1^{2n-2}, & 2^{2n-2}, & \ldots, & n^{2n-2}, & \pm\frac{1}{a^{2n-1}}f^{2n-1}(\tfrac{1}{2}l), & 0, & 0, & \ldots, & 0 \\
\sin ax, & -\frac{1}{2}\sin 2ax, & \ldots, & \pm\frac{1}{n}\sin nax, & fx, & \cos ax, & -\frac{1}{2^2}\cos 2ax, & \ldots, & \pm\frac{1}{n^2}\cos nax \\
0, & 0, & \ldots, & 0, & +\frac{1}{a^2}f''(\tfrac{1}{2}l), & 1, & 1, & \ldots, & 1 \\
0, & 0, & \ldots, & 0, & -\frac{1}{a^4}f^{\mathrm{iv}}(\tfrac{1}{2}l), & 1^2, & 2^2, & \ldots, & n^2 \\
0, & 0, & \ldots, & 0, & +\frac{1}{a^6}f^{\mathrm{vi}}(\tfrac{1}{2}l), & 1^4, & 2^4, & \ldots, & n^4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0, & 0, & \ldots, & 0, & \pm\frac{1}{a^{2n}}f^{2n}(\tfrac{1}{2}l), & 1^{2n-2}, & 2^{2n-2}, & \ldots, & n^{2n-2}
\end{vmatrix}. \qquad (142)$$

Expanding this with respect to the middle row, we have

$$fx - \sum_{r=1}^{r=n}\left[A_r \cos\frac{2\pi r}{l}x + B_r \sin\frac{2\pi r}{l}x\right].$$

We therefore derive the formula

$$fx - fy = \sum_{r=1}^{r=n}\left\{A_r\left[\cos\frac{2\pi r}{l}x - \cos\frac{2\pi r}{l}y\right]\right.$$

$$\left. + B_r\left[\sin\frac{2\pi r}{l}x - \sin\frac{2\pi r}{l}y\right]\right\} + R, \qquad (143)$$

wherein, when n is infinite,

$$A_r = 2 \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}}{r^2} {}_r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(\tfrac{1}{2}l),$$

$$B_r = 2 \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}}{r} {}_r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(\tfrac{1}{2}l).$$

Put $y = \frac{1}{2} l$, then $\cos \pi r = (-1)^r$, and we have for the constant term

$$A_0 = \sum_0^\infty \frac{1}{(2p+1)!} \left[\frac{l}{2}\right]^{2p} f^{2p}(\tfrac{1}{2}l) = \frac{1}{l} \int_0^l fx dx;$$

$$\therefore \qquad fx = A_0 + \sum_1^\infty \left[A_r \cos \frac{2\pi r}{l} x + B_r \sin \frac{2\pi r}{l} x \right], \tag{144}$$

wherein A_0, A_r, B_r have the values assigned above, and (144) is the standard form of Fourier's Theorem.

Exponential Forms.

39. We frequently meet such expressions as "If fx be a function developable in powers of e^x;" for example, the ordinary deduction of Abel's series is dependent upon this assumption; i. e.,

$$fx = A_0 + A_1 e^x + A_2 e^{2x} + \ldots + A_r e^{rx} + \ldots,$$

See Carr's Synopsis, p. 282. Again, in the same work, p. 394, we read, "Given that $F(x+a)$ can be expanded in powers of e^{-a}, then" Abel's formula for the definite integral of a complex function follows, based upon the assumption

$$F(x+a) = A_0 + A_1 e^{-a} + \ldots + A_r e^{-ra} + \ldots.$$

Our method shows clearly that these two expansions are impossible when we attempt them, by substitution in the *complete* composite, because when we attempt the expansion of fx according to the functions $e^{\pm rx}$ it is at once seen that these functions yield the body-determinant

$$\zeta^{\frac{1}{2}}(1, 2, \ldots, n),$$

for which, the value of the ratio $\rho(r, p)$ has been shown, § 32, to become infinite with n. We return to this form in § 42.

In Todhunter's Functions, p. 128, in an example determining the temperature of a homogeneous sphere placed in a medium of constant temperature,

we read, " We will also assume that as u is a function of t, it may be expanded in a series proceeding according to ascending powers of e^{-t}; this assumption may in some degree be justified by Burmann's Theorem. We assume, then, that u can be expressed in a series of the form

$$u = A_1 e^{-a_1{}^2 t} + A_2 e^{-a_2{}^2 t} + \ldots + A_r e^{-a_r{}^2 t} + \ldots . \text{"}$$

This expression, it is easy to see, is a possible one when the arbitraries have the values given by $a_r = r$; for then the functions $e^{-r^2 t}$ yield, in the composite, a body-determinant

$$\zeta^{\frac{1}{2}}(1^2, 2^2, \ldots, n^2).$$

We may, therefore, express the coefficients A_r in a converging infinite series in terms of Fourier's numbers and derivative forms of u.

40. Consider the expansion of fx in terms of the functions $e^{-r^2 ax}$, in which a is an arbitrary constant, and r takes successively the values 1, 2, 3, ... Making the substitutions in the composite and putting all of the arbitraries equal to z after differentiation, factoring all common factors and negative signs out of the body-determinant, we obtain

$$\begin{vmatrix} fx & , 1, & e^{a(z-x)}, & \frac{1}{2^2}e^{2^2a(z-x)}, & \ldots, & \frac{1}{n^2}e^{n^2a(z-x)}, & \frac{1}{(n+1)^2}e^{(n+1)^2a(z-x)} \\ fy & , 1, & e^{a(z-y)}, & \frac{1}{2_2}e^{2^2a(z-y)}, & \ldots, & \frac{1}{n^2}e^{n^2a(z-y)}, & \frac{1}{(n+1)^2}e^{(n+1)^2(z-y)} \\ -\frac{1}{a}f'z & , 0, & 1, & 1 & , \ldots, & 1 & , \quad 1 \\ +\frac{1}{a^2}f''z & , 0, & 1, & 2^2 & , \ldots, & n^2 & , \quad (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^n}{a^n}f^n z & , 0, & 1, & 2^{2n-2} & , \ldots, & n^{2n-2} & , \quad (n+1)^{2n-2} \\ \Phi(u) & , 0, & 0, & 0 & , \ldots, & 0 & , \quad 1 \end{vmatrix} = 0. \quad (145)$$

Expanding this, we obtain the formula

$$fx - fy = 2\sum_{r=1}^{r=n}\left(e^{r^2a(z-x)} - e^{r^2a(z-y)}\right)A_r + R, \quad (146)$$

wherein, for $n = \infty$,

$$A_r = \sum_{p=1}^{p=\infty}\frac{(-1)^r}{r^2 a^p}\,{}^r\mathfrak{F}_p\, f^p z,$$

a true series when the exponents of e are negative.

If $z = y = l$, we obtain

$$fx = A_0 + \sum_{r=1}^{r=n} A_r e^{r^2 a(l-x)} + R, \tag{147}$$

wherein the values of the constants are obvious.

In (147) put* $a = ib$, where i is the operator $\sqrt{-1}$; and also put

$$e^{r^2 ib(l-x)} = \cos r^2 b(l-x) + i \sin r^2 b(l-x);$$

whence

$$fx = (A + iB) + 2 \sum_{r=1}^{r=n} [\cos r^2 b(l-x) - i \sin r^2 b(l-x)](A_r + iB_r) + R,$$

in which, when $n = \infty$,

$$A_0 = f(l) + 2 \sum_{r=1}^{r=\infty} \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 b^{2p}} {}^r\delta_{2p} f^{2p}(l),$$

$$B_0 = 2 \sum_{r=1}^{r=\infty} \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 b^{2p-1}} {}^r\delta_{2p-1} f^{2p-1}(l),$$

$$A_r = \sum_{p=1}^{p=\infty} \frac{(-1)^r(-1)^{p+1}}{r^2 b^{2p}} {}^r\delta_{2p} f^{2p}(l),$$

$$B_r = \sum_{p=1}^{p=\infty} \frac{(-1)^r(-1)^{p+1}}{r^2 b^{2p-1}} {}^r\delta_{2p-1} f^{2p-1}(l).$$

By equating real and imaginary parts, if fx be a real function, then follow the series

$$fx = A_0 + 2 \sum_{r=1}^{r=n} [A_r \cos r^2 b(l-x) + B_r \sin r^2 b(l-x)] + R, \tag{148}$$

$$0 = B_0 + 2 \sum_{r=1}^{r=n} [B_r \cos r^2 b(l-x) - A_r \sin r^2 b(l-x)] + R.$$

Simpler forms follow when $b = 2\pi r/l$ or $b = 2\pi r/m$ and $l = 0$.

40. The function e^{-x^2} is a very important one in mathematical physics, it becomes unity when x vanishes, and vanishes itself an infinite number of times when $x = \pm \infty$.

The function

$$\varphi(x) = \frac{d^n e^{-x^2}}{dx^n} = H_n e^{-x^2},$$

* This substitution, introducing unreal quantities, is here permissible only as a matter of form. As yet we have not demonstrated the composite relation wherein complex quantities are involved. This demonstration will be given in another place for analytical functions of a complex variable; it differs but slightly from that for functions of a real variable.

is equal to the product of e^{-x^2} into a rational integral function of x of the nth degree, which we have indicated by H_n. These functions have been studied by M. Hermite, and according to custom we call them Hermite's functions (Laurent, Traité d'Analyse, T. V. p. 213).

Consider the expansion of fx in terms of the functions $e^{-ar^2x^2}$, a being an arbitrary positive constant, and r taking successively the values 1, 2, 3, ...

Put

$$y = e^{-ar^2x^2};$$

$$\therefore \qquad \log y = -ar^2x^2, \qquad \text{and} \qquad \frac{dy}{dx} = -2ar^2xy.$$

Differentiating $n + 1$ times, we obtain

$$\frac{d^{n+2}y}{dx^{n+2}} + 2ar^2x\frac{d^{n+1}y}{dx^{n+1}} + 2ar^2(n+1)\frac{d^ny}{dx^n} = 0.$$

Dividing through by $e^{-ar^2x^2}$, we obtain the following relation between three consecutive functions of Hermite:

$$H_{n+2} + 2ar_2x\,H_{n+1} + 2ar^2(n+1)\,H_n = 0;$$

and if $x = 0$, we have

$$^0H_{n+2} = -2ar^2(n+1)\,^0H_n.$$

Since $^0H_1 = 0$, we have $^0H_{2n-1} = 0$, and since $^0H_2 = -2ar^2$ it is easy to see that

$$^0H_{2n} = (-1)^{n+1}\,2^na^nr^{2n}\,.\,3\,.\,5\,.\,7\,\ldots\,(2n-1)$$

$$= (-1)^{n+1}\,a^nr^{2n}\,\frac{(2n)!}{n!}.$$

Since on factoring the common factors out of the body-determinant we divide r^2 out of the rth column, we may regard the expansion of fx as effected according to the functions $r^{-2}\,e^{-ar^2x^2}$.

Making the proper substitutions in the composite, and clearing the body-determinant of all factors, and putting the arbitraries equal to zero after differentiation, remembering that the composite in this case can contain no odd derivative forms, we have

$$\begin{vmatrix} fx & , 1, & e^{-ax^2}, & \frac{1}{2^2}e^{-a2^2x^2}, & \ldots, & \frac{1}{n^2}e^{-an^2x^2}, & \frac{1}{(n+1)^2}e^{-a(n+1)^2x^2} \\ fy & , 1, & e^{-ay^2}, & \frac{1}{2^2}e^{-a2^2y^2}, & \ldots, & \frac{1}{n^2}e^{-an^2y^2}, & \frac{1}{(n+1)^2}e^{-a(n+1)^2y^2} \\ \frac{-1!}{a2!}f''0 & , 0, & 1, & 1, & \ldots, & 1, & 1 \\ \frac{+2!}{a^2 4!}f^{\text{iv}}0 & , 0, & 1, & 2^2, & \ldots, & n^2, & (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^{n+1}\,n!}{a^n\,(2n)!}f^{2n}0 & , 0, & 1, & 2^{2n-2}, & \ldots, & n^{2n-2}, & (n+1)^{2n-2} \\ \Phi(u) & , 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0\,. \quad (149)$$

Whence follows

$$fx - fy = \sum_{r=1}^{r=n} A_r\,(e^{-ar^2x^2} - e^{-ar^2y^2}) + R \qquad (150)$$

wherein when $n = \infty$, we have

$$A_r = 2\sum_{p=1}^{p=\infty} \frac{(-1)^r}{r^2a^p}\,{}^r\mathfrak{F}_p\,\frac{p!}{(2p)!}f^{2p}(0)\,.$$

If $y = 0$, then

$$fx = A_0 + \sum_{r=1}^{r=n} A_r\,e^{-ar^2x^2} + R\,, \qquad (151)$$

in which the value of A_0, for $n = \infty$, is obvious.

If $y = \infty$, then

$$fx - f\infty = 2\sum_{r=1}^{\infty} e^{-ar^2x^2}\sum_{p=1}^{\infty}\frac{(-1)^r}{r^2a^p}\,{}^r\mathfrak{F}_p\,\frac{p!}{(2p)!}f^{2p}(0)$$

$$= \int_{\infty}^{x} f'x\,dx\,;$$

whence

$$\int_{x}^{\infty}\psi x\,dx = 2\sum_{r=1}^{\infty} e^{-ar^2x^2}\sum_{p=1}^{\infty}\frac{(-1)^{r+1}}{r^2a^p}\,{}^r\mathfrak{F}_p\,\frac{p!}{(2p)!}\psi^{2p-1}(0)\,,$$

and if $x = 0$,

$$\int_{0}^{\infty}\psi x\,dx = \sum_{0}^{\infty}\frac{1}{a^p(2p+1)!}\,\frac{p!}{(2p)!}\,\psi^{2p-1}(0)\,,$$

provided fx be a function developable as above.

In (150) put $a = ib$, and

$$e^{-ibr^2x^2} = \cos br^2x^2 - i \sin br^2x^2 .$$

Then

$$fx = A_0 + iB_0 + \sum_{r=1}^{n} (\cos br^2x^2 - i \sin br^2x^2)(A_r + iB_r) + R ,$$

wherein, when $n = \infty$,

$$A_0 = \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^p}{b^{2p}(4p+1)!} \frac{(2p)!}{(4p)!} f^{4p}(0) ,$$

$$B_0 = \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{b^{2p-1}(4p-1)!} \frac{(2p-1)!}{(4p-2)!} f^{4p-2}(0) ,$$

$$A_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^r(-1)^p}{b^{2p}r^2} \frac{(2p)!}{(4p)!} {}^r\delta_{2p} f^{4p}(0) ,$$

$$B_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^r(-1)^{p+1}}{b^{2p-1}r^2} \frac{(2p-1)!}{(4p-2)!} {}^r\delta_{2p-1} f^{4p-2}(0) .$$

Equating real and unreal parts, we have, if fx is real,

$$fx = A_0 + \sum_{1}^{n} (A_r \cos br^2x^2 + B_r \sin br^2x^2) + R , \tag{152}$$

$$0 = B_0 + \sum_{1}^{n} (B_r \cos br^2x^2 - A_r \sin br^2x^2) + R .$$

41. Consider the function x^2a^x, we have

$$\frac{d^n}{dx^n}(x^2a^x) = [x^2c^n + 2nxc^{n-1} + n(n-1)c^{n-2}]a^x ,$$

wherein $c = \log a$ and $a = e^c$.

If $x = 0$, then

$$\left[\frac{d^n}{dx^n}(x^2a^x)\right]_0 = n(n-1)c^{n-2} .$$

Let $c = -ar^2 = \log a$, so that $a = e^{-ar^2}$. Consider the expansion of fx in terms of the functions

$$x^2 e^{-ar^2x} ,$$

which for brevity in printing the determinant we symbolize by $\varphi_x(r)$.

Noticing that these functions do not contain the first power of x, it follows that the composite will not contain the first derivative row. Making the substitutions and factoring the body-determinant to its simplest form, after putting the arbitraries equal to zero after differentiation and observing the

formula

$$\left[\frac{d^n}{dx^n}\,(x^2e^{-ar^2x})\right]_0 = (-1)^{n-2}\,n(n-1)\,a^{n-2}\,r^{2n-4}\,,$$

we have

$$\begin{vmatrix} fx & ,\,1\,, & \varphi_x(1)\,, & \varphi_x(2)\,, & \ldots, & \varphi_x(n)\,, & \varphi_x(n+1) \\ fy & ,\,1\,, & \varphi_y(1)\,, & \varphi_y(2)\,, & \ldots, & \varphi_y(n)\,, & \varphi_y(n+1) \\ +\frac{1}{1\,.\,2}f''0 & ,\,0\,, & 1 & ,\,1 & ,\ldots, & 1 & ,\quad 1 \\ -\frac{1}{a}\frac{1}{2\,.\,3}f'''0 & ,\,0\,, & 1 & ,\,2^2 & ,\ldots, & n^2 & ,\,(n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^n}{a^{n-2}n(n-1)}f^n0, & 0\,, & 1 & ,\,2^{2n-4} & ,\ldots, & n^{2n-4} & ,\,(n+1)^{2n-4} \\ \Phi(n) & ,\,0\,, & 0 & ,\,0 & ,\ldots, & 0 & ,\quad 1 \end{vmatrix} = 0. \qquad (153)$$

Expanding this we get the formula

$$fx - fy = \sum_{r=1}^{n} A_r\,(x^2e^{-ar^2x} - y^2e^{-ar^2y}) + R\,, \qquad (154)$$

wherein, when $n = \infty$,

$$A_r = 2\sum_{p=2}^{\infty}\frac{(-1)^{r+1}}{a^{p-2}\,p(p-1)}\,{}^r\mathfrak{F}_{p-1}\,f^p(0)\,.$$

If we put $y = 0$ and $a = ib$ in (154), we have, after equating real parts, if fx is a real function,

$$fx = f0 + x^2\sum_{r=1}^{n}(A'_r\cos br^2x + B'_r\sin br^2x) + R\,, \qquad (155)$$

wherein, when $n = \infty$,

$$A'_r = 2\sum_{p=1}^{\infty}\frac{(-1)^{r+1}(-1)^{p+1}}{a^{2p-2}\,2p(2p-1)}\,{}^r\mathfrak{F}_{2p-1}\,f^{2p}(0)\,,$$

$$B'_r = 2\sum_{p=1}^{\infty}\frac{(-1)^{r+1}(-1)^p}{a^{2p-1}\,2p(2p+1)}\,{}^r\mathfrak{F}_{2p}\,f^{2p+1}(0)\,.$$

42. Before leaving these examples of exponential forms, let us return to the form

$$fx = \sum_{r=0}^{\infty} A_r\,e^{rax}\,,$$

when the exponent of e is a real quantity. We have seen how any expansion in powers of e leads to an expansion of a complex harmonic function, but none of these complex harmonics got from exponential series have led to Fourier's

form, and only the expansion above can lead to that well established theorem. The expansion must therefore be possible, and in casting about for an explanation of the seeming difficulty I have sought satisfaction in the following course of reasoning.*

Consider the expansion of fx according to the functions e^{-rax}, in which a and x are positive and r successively the integers 1, 2, 3, ...

If this expansion be possible, then we must have

$$fx = \sum_0^\infty A_r\, e^{-rax}\,,$$

wherein the coefficients A_r are independent of x. This being so, we have, after n differentiations,

$$f^n x = \sum_1^\infty A_r\,(-1)^n\, r^n a^n e^{-rax}\,.$$

Therefore, since the second member vanishes when x is infinite, we must have $f^n\infty = 0$. Hence we observe that any one of the derivative rows in the composite may be made to vanish, and therefore the determinant, by putting $x = \infty$ after differentiation. Hence, by exactly the same method of reasoning as that employed in establishing the theorem of § 4, it follows that we may omit, in this case, from the composite any derivative row, provided we include infinity among the arbitraries, between the greatest and least of which lies u. We may, therefore, omit all of the odd derivative rows or all of the even derivative rows. But at no time can two consecutive rows be omitted.

Making the substitutions in the composite, containing only even derivative rows, and putting the arbitraries after differentiation equal to l, after factoring to simplest form, we have

$$\begin{vmatrix} fx & , 1, & e^{a(l-x)}, & \frac{1}{2^2}e^{2a(l-x)}, & \ldots, & \frac{1}{n^2}e^{na(l-x)}, & \frac{1}{(n+1)^2}e^{(n+1)a(l-x)} \\ f(l) & , 1, & 1, & \frac{1}{2^2}, & \ldots, & \frac{1}{n^2}, & \frac{1}{(n+1)^2} \\ \frac{1}{a^2}f''(l) & , 0, & 1, & 1, & \ldots, & 1, & 1 \\ \frac{1}{a^4}f^{\text{iv}}(l) & , 0, & 1, & 2^2, & \ldots, & n^2, & (n+1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a^{2n}}f^{2n}(l), & 0, & 1, & 2^{2n-2}, & \ldots, & n^{2n-2}, & (n+1)^{2n-2} \\ \Phi(u) & , 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0\,. \qquad (156)$$

* Which is far from being satisfactory. June, 1893.

So that

$$fx = A_0 + \sum_{r=1}^{n} A_r e^{ra(l-x)} + R, \tag{157}$$

wherein, for $n = \infty$,

$$A_0 = f(l) - 2 \sum_{r=1}^{r=\infty} \sum_{p=1}^{p=\infty} \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 a^{2p}} {}^r\mathfrak{F}_p f^{2p}(l),$$

$$A_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^{r+1}(-1)^{p+1}}{r^2 a^{2p}} {}^r\mathfrak{F}_p f^{2p}(l).$$

In like manner we deduce from the composite in which we retain the odd derivative rows,

$$fx = B_0 + \sum_{r=1}^{n} B_r e^{ra(l-x)} + R, \tag{158}$$

in which, for $n = \infty$,

$$B_0 = fl - 2 \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{(-1)^{r+1}(-1)^p}{a^{2p-1} r} {}^r\mathfrak{F}_p f^{2p-1}(l),$$

$$B_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^{r+1}(-1)^p}{r a^{2p-1}} {}^r\mathfrak{F}_p f^{2p-1}(l).$$

If in (157) and (158) we put $a = ib$, and $b = 2\pi/l$, also

$$e^{rib(l-x)} = \cos rb(l-x) + i \sin rb(l-x),$$

we have by equating reals in (157),

$$fx = A'_0 + \sum_{r=1}^{\infty} A'_r \cos \frac{2\pi r}{l} x, \tag{159}$$

wherein

$$A'_0 = \frac{1}{l} \int_{\frac{1}{2}l}^{\frac{3}{2}l} fx dx, \quad A'_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^{r+1}}{r^2} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p} f^{2p}(l),$$

which is the same as (134).

Equating imaginaries in (158), we have

$$fx = fl + \sum_{r=1}^{\infty} B'_r \sin \frac{2\pi r}{l} x,$$

wherein

$$B'_r = 2 \sum_{p=1}^{\infty} \frac{(-1)^{r+1}}{r} {}^r\mathfrak{F}_p \left[\frac{l}{2\pi}\right]^{2p-1} f^{2p-1}(l),$$

which is the same form as (138).

Hyperbolic Functions.

43. If it be possible to expand a function in terms of the hyperbolic sines and cosines, we may obtain the form of that expansion in a manner almost identical with that employed for the circular functions.

Thus, consider the form of the expansion in terms of the functions

$$\frac{1}{r^2}\cosh rax\,.$$

Putting $x = z$ after differentiation, and observing that the odd rows are to be left out, because cosh is an even function, we deduce from the composite

$$fx - fy = \sum_{r=1}^{n} \frac{A_r}{\cosh raz}(\cosh rax - \cosh ray) + R\,, \tag{160}$$

wherein, if $n = \infty$,

$$A_r = 2\sum_{p=1}^{\infty} \frac{(-1)^{r+1}(-1)^{p+1}}{a^{2p}r^2}\,{}^r\mathfrak{F}_p\,f^{2p}z\,.$$

Put $y = z = 0$; then, if $a = 1/l$,

$$fx = A'_0 + \sum_{r=1}^{n} A'_r \cosh\frac{rx}{l} + R\,, \tag{161}$$

wherein, if infinite values of n be permissible,

$$A'_0 = \sum_{0}^{\infty} \frac{(-1)^p\, l^{2p}}{(2p+1)!} f^{2p}(0)\,, \qquad A'_r = \sum_{p=1}^{\infty} \frac{(-1)^{r+1}}{r^2}(-1)^{p+1}\, l^{2p}\, {}^r\mathfrak{F}_p\, f^{2p}(0).$$

Again, in (160) put $a = i\pi/l$ and $z = y = l$, then $\cosh ri\pi = (-1)^r$ and $a^{2p} = (-1)^p(\pi/l)^{2p}$;

$$\therefore \qquad fx = A_0'' + \sum_{r=1}^{n} A_r'' \cosh\frac{ri\pi}{l}x + R\,, \tag{162}$$

and

$$A_0'' = \frac{1}{l}\int_{\frac{1}{2}l}^{\frac{3}{2}l} fx\,dx\,, \qquad A_r'' = 2\sum_{p=1}^{\infty}\frac{1}{r^2}\,{}^r\mathfrak{F}_p\left[\frac{l}{2\pi}\right]^{2p} f^{2p}(l)\,.$$

In exactly the same manner we deduce the formula

$$fx - fy = \sum_{r=1}^{n} \frac{B_r}{\cosh raz}(\sinh rax - \sinh ray) + R\,, \tag{163}$$

wherein, if $n = \infty$,

$$B_r = 2\sum_{p=1}^{\infty}\frac{(-1)^{r+1}(-1)^{p+1}}{ra^{2p-1}}\,{}^r\mathfrak{F}_p\, f^{2p-1}z\,.$$

From which we deduce a formula similar to (161) by putting $y = z = 0$ and $a = 1/l$.

Thus

$$fx = f(0) + \sum_{r=1}^{n} B_r' \sinh \frac{rx}{l} + R. \tag{164}$$

If fx is a function which vanishes with x, then $f(0) = 0$. By addition of (161) and (164), we have

$$fx = C_0 + \sum_{r=1}^{n} \left[C_r' \cosh \frac{rx}{l} + C_r'' \sinh \frac{rx}{l} \right] + R, \tag{165}$$

a formula analagous to Fourier's theorem, in which the values of the constant coefficients, when $n = \infty$, are at once obvious from the above.

Bessel's Functions.

44. Consider the expansion of fx in terms of the functions

$$J_{2m}(rax) = \sum_{l=0}^{\infty} \frac{(-1)^l (ra)^{2m+2l}}{l!\,(2m+l)!} \left[\frac{x}{2}\right],$$

in which m is any positive integer including zero. This function is an even function containing no power of x lower than $2m$. The composite will, by § 4, contain no odd derivative rows, nor will it contain any even derivative row of lower order than $2m$.

Substitute in the composite the functions

$$\frac{1}{r^{2m}} J_{2m}(rax),$$

and put $x = 0$ after differentiation. Factoring to simplest shape, we have

$$\begin{vmatrix} & fx, & 1, & J_{2m}(ax), & \frac{1}{2^{2m}} J_{2m}(2ax), & \ldots, & \frac{1}{n^{2m}} J_{2m}(nax), & \frac{1}{(n+1)^{2m}} J_{2m}(\overline{n+1}\,ax) \\ & fy, & 1, & J_{2m}(ay), & \frac{1}{2^{2m}} J_{2m}(2ay), & \ldots, & \frac{1}{n^{2m}} J_{2m}(nay), & \frac{1}{(n+1)^{2m}} J_{2m}(\overline{n+1}\,ay) \\ \left[\frac{2}{a}\right]^{2m} & f^{2m}(0), & 0, & 1, & 1, & \ldots, & 1, & 1 \\ -\frac{1!\,(2m+1)!}{(2m+2)!}\left[\frac{2}{a}\right]^{2m+2} & f^{2m+2}(0), & 0, & 1, & 2^2, & \ldots, & n^2, & (n+1)^2 \\ +\frac{2!\,(2m+2)!}{(2m+4)!}\left[\frac{2}{a}\right]^{2m+4} & f^{2m+4}(0), & 0, & 1, & 2^4, & \ldots, & n^4, & (n+1)^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{n-1}\frac{(n-1)!\,(2m+n-1)!}{[2m+2(n-1)]!}\left[\frac{2}{a}\right]^{2(m+n-1)} & f^{2(m+n-1)}(0), & 0, & 1, & 2^{2n-2}, & \ldots, & n^{2n-2}, & (n+1)^{2n-2} \\ & \phi(u), & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \tag{167}$$

Expanding this we obtain

$$fx - fy = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r^{2m}} A_r \left[J_{2m}(rax) - J_{2m}(ray) \right] + R, \tag{168}$$

wherein, if $n = \infty$,

$$A_r = 2 \sum_{p=1}^{\infty} \frac{(p-1)!\,(2m+p-1)!}{(2m+2p-1)!} \left[\frac{2}{a}\right]^{2(m+p-1)} {}^r\mathfrak{Z}_p\,.\,f^{2(m+p-1)}(0).$$

If $y = 0$ and fx a function vanishing with x, then for $m > 0$ we have

$$fx = \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r^{2m}} A_r J_{2m}(rax) + R. \tag{169}$$

In exactly the same manner we deduce the expansion in terms of any odd function $J_{2m-1}(rax)$, there being no difference in the result, save that we must change the even derivatives to odd ones.

So that we have

$$fx - fy = \sum_{r=1}^{n} A_r \left[J_m(rax) - J_m(ray) \right], \tag{170}$$

wherein, if $n = \infty$,

$$A_r = 2 \sum_{p=1}^{\infty} \frac{(p-1)!\,(m+p-1)!}{(m+2p-1)!} \left[\frac{2}{a}\right]^{m+2(p-1)} \frac{(-1)^{r+1}}{r^m} {}^r\mathfrak{Z}_p\,.\,f^{m+2(p-1)}(0).$$

If $m > 0$ there will be no absolute term, since $J_0(ray)$ is the only function which has a constant term when we put $y = 0$.

In (170) put $m = 0$ and $a = 2$, also $y = 0$; then, since $J_0(ray) = -1$, we have

$$fx = B_0 + \sum_{r=1}^{n} B_r J_0(2rx) + R, \tag{171}$$

wherein, for $n = \infty$,

$$B_0 = f(0) + 2 \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} (-1)^r \frac{(p!)^2}{(2p)!} {}^r\mathfrak{Z}_p f^{2p}(0),$$

$$B_r = 2 \sum_{p=1}^{\infty} (-1)^{r+1} \frac{(p!)^2}{(2p)!} {}^r\mathfrak{Z}_p f^{2p}(0).$$

This last result (171) is the remarkable theorem due to Schlömilch (Tod-

hunter's Functions, p. 336). The value of the coefficient as determined by Schlömilch, however, is

$$B_r = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\left\{ f(0) + u\int_0^1 \frac{f'(u\xi)\,d\xi}{\sqrt{(1-\xi^2)}}\right\}\cos 2ru du\,,$$

for every value of u except zero, when we must add $2f(0)$.

Other interesting forms may be obtained from (170) through suppositions regarding the arbitrary constants.

UNIVERSITY OF VIRGINIA, *Jan.* 25, 1892.

FORSYTH'S THEORY OF FUNCTIONS OF A COMPLEX VARIABLE.*

By Prof. W. H. Echols, Charlottesville, Va.

There is just out of the Cambridge University Press a book which will mark an epoch in mathematics for English and American students. It puts the English speaking mathematical student in possession, for the first time in his own tongue, of those splendid developments in analysis which have been created in the last few years, and which are being created to-day. Heretofore, he has had to go to the journals, special memoirs, treatises in French and in German, or to Germany itself, to get an insight of what has been, and is being done, in this far reaching living field of research.

This splendid book will be most heartily and gratefully welcomed by the American student, who can now read in the pure, strong, and simple English of Forsyth these beautiful theorems, whose discovery has made immortal the genius of the master mathematicians of our century. A marked feature of the work is the complete reference in the body of the text, as well as in the foot-notes, to authors and their works; and one cannot fail to be impressed by the marked absence from among them of English names.

The Cambridge, or any other, Press has turned out no more beautifully made book than this royal octavo volume of six hundred and eighty pages. It is a model of the art of mathematical printing, and seems to be as free from printer's error as only the careful English printer can be.

A glance at the short preface outlines clearly the scope of the work. It is a marshalling of the main results of the three distinct methods of investigation of Cauchy, Weierstrass, and Riemann. The general method which is adapted is an attempt to give a consecutive account of what may be fairly deemed the principal branches of the whole subject, and is not limited so that it may conform to any single one of the three principal independent methods; ideas and processes derived from different processes have been combined where it has been convenient. In this respect this book is unique; for while there is no dearth of treatises in French and in German on this subject, they, for the most part, expound the processes based on some single method, or they deal with the discussion of some particular branch of the theory.

There are five natural divisions of the book.

The first part, consisting of Chapters I–VII, contains the theory of uniform (single-valued) functions; the discussion is based upon power series, initially connected with Cauchy's theorems in integration.

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* Theory of Functions of a Complex Variable. By A. R. Forsyth. Macmillan & Co., New York. $8.50.

The second part, consisting of Chapters VIII–XIII, contains the theory of multiform functions and of uniform periodic functions which are derived through the inversion of integrals of algebraic functions.

The third part, consisting of Chapters XIV–XVIII, contains the development of the theory of functions according to the method initiated by Riemann in his memoirs.

The fourth part, consisting of Chapters XIX and XX, treats of conformal representation.

The fifth part, consisting of Chapters XXI and XXII, contains an introduction to the theory of Fuchsian or automorphic functions, based on the researches of Poincaré and Klein.

The author closes his preface by remarking, "My aim has been to produce a book that will assist mathematicians in acquiring a knowledge of the theory of functions: in proportion as it may prove of real service to them, will be my reward." His reward is assured. The American student is already deeply in debt to Forsyth, and we can but extend to him our grateful thanks in profound recognition of this new obligation he has placed upon us.

Macmillan & Co. announce a new treatise on the Theory of Functions by Professors Harkness and Morley which we await impatiently, and hope sincerely will prove a valuable addition to our subject matter.

It remains now for some one to put into English words Dini's Theory of Functions of a Real Variable, for us to possess a continuous exposition of Function Theory from Chrystal's Algebra through Forsyth's masterly work, which will enable the American mathematician to read up to the modern developments of this rapidly growing subject in his own language.

A STUDY OF CERTAIN SPECIAL CASES OF THE HYPERGEOMETRIC DIFFERENTIAL EQUATION.

By Mr. James Harrington Boyd, Chicago, Ill.

Preface.

The following paper presents the results of a study of certain cases of the hypergeometric differential equation made at Göttingen, under the direction of Professor Klein. The cases are those in which the hypergeometric differential equation has a single algebraic solution (Schwarz, Crelle, 75, § 1).

The method is the geometric one used by Klein in his lectures on linear differential equations, and in his paper "*Ueber die Nullstellen der hypergeometrischen Reihe*," Math. Annalen, Bd. 37, S. 580. The solutions are regarded as functions defined, after the manner of Riemann, by the requirement that the conformal representations* which they determine be certain generalized triangles. These triangles are made the starting point of the discussion, and are classified geometrically, and the classification of the integrals of the equation is deduced from that of the triangles.

Owing to the fact that no exposition of Prof. Klein's theory of the linear differential equation of the second order has yet been published, it has been deemed best to prefix to the paper a pretty full presentation of his peculiar methods and notation, drawn from his lectures on differential equations, Winter semester, 1890–1891.

The discussion of Case α), Arts. 24 foll. (excepting for integral values of λ, μ, ν) and Figs. 1–13, 18, 28, 36–46 are due to Klein (lectures on differential equations).

On the other hand, the discussion of Cases β) and γ), and Figs. 14–17, 19–27, 29–35, 47–53 are due to the author.

N. B.—The shaded parts of the figures in the η-plane are the conformal representations of the positive half of the z-plane; the blank portions, of the negative half.

Introduction.

§ 1. *The Hypergeometric Function.*

If we subject the Abelian integral

$$\varphi_{ik} = \int_i^k (za)^{\alpha}(zb)^{\beta}(zc)^{\gamma}(zd)^{\delta}(zdz) \tag{1}$$

* See below, § 4, Art. 5.

(in which

$$z = \frac{z_1}{z_2}, \quad p = \frac{p_1}{p_2}, \quad a = \frac{a_1}{a_2}, \quad \ldots;$$

$$(za) = \begin{vmatrix} z_1 & a_1 \\ z_2 & a_2 \end{vmatrix}, \quad \ldots, \quad (zdz) = \begin{vmatrix} z_1 & dz_1 \\ z_2 & dz_2 \end{vmatrix};$$

$$\alpha + \beta + \gamma + \delta = -2;$$

$a, b, c, d, \alpha, \beta, \gamma, \delta$ are complex numbers; and $z = a, b, c, d$ are *branch points** of the integral φ_{ik})† to the system of linear substitutions

$$\left.\begin{array}{l} z'_1 = p_1 z_1 + p_2 z_2, \quad z'_2 = q_1 z_1 + q_2 z_2; \\ a'_1 = p_1 a_1 + p_2 a_2, \quad a'_2 = q_1 a_1 + q_2 a_2; \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{2}$$

we shall obtain

$$\varphi_{ik} = (pq)^2 \int_i^k (z'a')^\alpha (z'b')^\beta (z'c')^\gamma (z'd')^\delta (z'dz'). \tag{3}$$

The notation leading up to (3) will be retained throughout this paper.

If we specialize the substitutions in (2), so that the points a', b', c' become respectively $0, \infty, 1$,‡ we shall obtain

$$\Phi_{ik} = K \int_i^k z'^{\alpha}_1 z'^{\beta}_2 (z'_1 - z'_2)^\gamma (z'\lambda')^\delta (z'dz'); \tag{4}$$

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* Forsyth's Theory of Functions, p. 15.

† When either i, k, or both, are points of discontinuity of the integral, φ_{ik} is to be regarded as defined by the equation

$$\varphi_{ik} = \frac{\int^{\overline{abab}} \varphi(z)dz}{(1 - e^{2\pi i\alpha})(1 - e^{2\pi i\beta})},$$

where

$$\int^{\overline{abab}} \varphi(z)dz$$

is the result of integrating φ on a closed path which encircles each of the points a, b twice, but the second time in the opposite sense from the first time. Jordan, Cours d'Analyse, Vol. III, p. 241.

‡ By permuting a', b', c' we may transform a', b', c' into $0, \infty, 1$ in six different ways; hence will arise the six different classes of hypergeometric series (see Forsyth's Differential Equations § 119).

or, written non-homogeneously,

$$\Phi_{ik} = K\int_i^k z^\alpha (z-1)^\gamma (z-\lambda)^\delta \, dz\,, \tag{5}$$

where

$$\lambda = \frac{(bc)(ad)}{(ac)(bd)},$$

and K is a constant different from zero.

As is well known, we may write one of the integrals in (5)

$$\Phi_{01} = K' \,.\, B(\alpha+1, \gamma+1) \,.\, F\left[-\delta, \alpha+1, \alpha+\gamma+2, \frac{1}{\lambda}\right],^* \tag{6}$$

where K' is a constant, B is the Eulerian integral of the second species, and F a Gaussian hypergeometric series.

We define a *hypergeometric function* by the equation

$$H(\lambda) = C'\lambda^m (1-\lambda)^n \,.\, \omega\,, \tag{7}$$

where C' is a constant distinct from zero, and

$$\omega = \int_i^k z^\alpha (z-1)^\beta (z-\lambda)^\gamma \, dz\,;\dagger$$

or in accordance with (6),

$$\omega = K \,.\, F(l, m, n, \lambda)\,,$$

or a hypergeometric series whose argument, instead of λ, is either

$$\frac{1}{\lambda}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda-1}{\lambda}, \quad \text{or } \frac{\lambda}{\lambda-1}.$$

The elements l, m, n are constants involving α, β, γ.

§ 2. *The Hypergeometric Differential Equation.*

By hypergeometric differential equation we mean the differential equation which the function H defined by (7) will satisfy.

One can readily show that ω will satisfy the differential equation

$$\frac{d^2\omega}{d\lambda^2} - \frac{(\alpha+\beta+2\gamma)\lambda - (\alpha+\gamma)}{\lambda^2-\lambda} \cdot \frac{d\omega}{d\lambda} + \frac{(\alpha+\beta+\gamma+1)\gamma}{\lambda^2-\lambda} \,.\, \omega = 0\,. \tag{8}$$

* We obtain (6) by expanding $\left[1-\frac{z}{\lambda}\right]^\delta$, $(|\lambda| > 1)$, by the binomial theorem, and integrating between 0 and 1, and expressing the result in terms of Eulerian integrals.

† These integrals are called hypergeometric integrals.

From (7)

$$\omega = \frac{H(\lambda)}{C'\lambda^m(1-\lambda)^n},$$

whence, substituting in (8),

$$\frac{d^2H}{d\lambda^2} + \frac{A\lambda + B}{\lambda^2 - \lambda}\cdot\frac{dH}{d\lambda} + \frac{C\lambda^2 + D\lambda + E}{(\lambda^2-\lambda)^2}\cdot H = 0, \qquad (9)$$

where A, B, C, D, E are constants involving m, n, α, β, γ and C'.

§ 3. *The Riemann P-Function.*

The relation of the *Riemann P-function* to the *hypergeometric function*, and hence to the *hypergeometric differential equation*.*

1. Riemann represented by the symbol

$$P\begin{Bmatrix} a & b & c & \\ \lambda' & \mu' & \nu' & x \\ \lambda'' & \mu'' & \nu'' & \end{Bmatrix}, \qquad (10)$$

a binary *shear*† of functions having the following characteristics :—

1° Two functions of the shear have about the point a developments of the form

$$P_a^{\lambda'} = (x-a)^{\lambda'}\cdot p_1(x-a), \quad P_a^{\lambda''} = (x-a)^{\lambda''}\cdot p_2(x-a),$$

respectively, and every other function of the shear, a development of the form

$$c'_a P_a^{\lambda'} + c_a'' P_a^{\lambda''},$$

where the p's are series in ascending powers of $x - a$, whose constant terms do not vanish, and the c_a's are constants. The functions of the shear are similarly related in the neighborhood of the point b to the indices μ', μ'', and in the neighborhood of the point c to the indices ν', ν''. The a, b, c are in general complex numbers, and represent branch points of the functions, while x is any point in the plane within the regions of convergence about a, b, c.

2° $\lambda' + \lambda'' + \mu' + \mu'' + \nu' + \nu'' = 1$, and $\lambda' - \lambda''$, $\mu' - \mu''$, $\nu' - \nu''$ are not integral.

3° All functions of the shear have about every other point x_0 developments in integral powers of $x - x_0$.

It can be shown that these characteristics constitute a complete definition of the binary shear of functions.

* Riemann's collected works, IV.

† Compare the German *Schaar*.

2. As in § 1, we can by linear substitutions transform a, b, c (d remaining fixed) into the points 0, ∞, 1, respectively, so that

$$P\left\{\begin{matrix} a & b & c & \\ \lambda' & \mu' & \nu' & d \\ \lambda'' & \mu'' & \nu'' & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & \infty & 1 & \\ \lambda' & \mu' & \nu' & \lambda \\ \lambda'' & \mu'' & \nu'' & \end{matrix}\right\}, \tag{11}$$

where

$$\lambda = \frac{(ad)\,(bc)}{(bd)\,(ac)}.^*$$

P is, therefore, a function of the cross-ratio, only, of a, b, c, d.

3. From the definition of P it at once follows that

$$\lambda^m(1-\lambda)^n P\left\{\begin{matrix} 0 & \infty & 1 & \\ \lambda' & \mu' & \nu' & \lambda \\ \lambda'' & \mu'' & \nu'' & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & \infty & 1 & \\ \lambda'+m & \mu'+l & \nu'+n & \lambda \\ \lambda''+m & \mu''+l & \nu''+n & \end{matrix}\right\}, \tag{12}$$

where $l+m+n=0$.

4. We define a function Π_{ik} by the equation

$$\Pi_{ik} = M\varphi_{ik} = M\int_i^k (za)^\alpha\,(zb)^\beta\,(zc)^\gamma\,(zd)^\delta\,(zdz)\,, \tag{13}$$

where

$$M = (ab)^{-\frac{1}{2}(\alpha+\beta+1)}\,.\,(ac)^{-\frac{1}{2}(\alpha+\gamma+1)}\,.\,(ad)^{-\frac{1}{2}(\alpha+\delta+1)}\,.\,(bc)^{-\frac{1}{2}(\beta+\gamma+1)}\,.$$

$$(bd)^{-\frac{1}{2}(\beta+\delta+1)}\,.\,(cd)^{-\frac{1}{2}(\gamma+\delta+1)}\,.\;[(ad)\,(bc)]^p\,.\,[(bd)\,(ac)]^q\,.\,[(cd)\,(ab)]^r\,,$$

and $p+q+r=\frac{1}{2}$.

By the substitutions which transformed (3) into (4)

$$\Pi_{ik} = \lambda^{p+\frac{1}{2}(\delta-\alpha-1)}(\lambda-1)^{r-\frac{1}{2}(\gamma+\delta+1)}\int_i^k z_1^{\alpha}\,z_2^{\beta}\,(z_1-z_2)^\gamma\left[z_2-\frac{z_1}{\lambda}\right]^\delta(zdz)\,. \tag{14}$$

We select a branch Π_{ad} of Π_{ik} and study its behavior when the two points a and d are brought into coincidence. We notice that M contains the factor $(ad)^{p-\frac{1}{2}(\alpha+\delta+1)}$. As a approaches d the value of the integral in (13) will approach zero as $C'(ad)^{\alpha+\delta+1}_{a\doteq d}$, where C' is a constant; therefore Π_{ad} will approach zero as $K'(ad)^{p+\frac{1}{2}(\alpha+\delta+1)}_{a\doteq d}$, where K' is a constant different from zero. Also,

* By permuting a, b, c in all possible ways we can transform a, b, c into 0, ∞, 1 in six different ways; accordingly instead of λ in (11) we may put λ, $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{\lambda-1}$, or $\frac{\lambda-1}{\lambda}$.

as $a \doteq d$ the value of the integral H_{bc} approaches zero as $K''(ad)_{a \doteq d}^{p - \frac{1}{2}(\alpha + \delta + 1)}$, K'' being a constant different from zero. That is to say, in the region about the zero point (i. e. the point $a \doteq d$) we can develop these two H's in the form

$$H_{ad} = \lambda^{p + \frac{1}{2}(\alpha + \delta + 1)} \cdot p_1(\lambda),$$

$$H_{bc} = \lambda^{p - \frac{1}{2}(\alpha + \delta + 1)} \cdot p_2(\lambda),$$

since $\lambda \doteq 0$ as $a \doteq d$.

Thus, the exponents which characterize H for $\lambda = 0$ are determined, and we find that H_{ad} and H_{bc} coincide with the branches $P_0^{\lambda'}$ and $P_0^{\lambda''}$ of the P-function for $\lambda = 0$, when we put

$$\lambda' = p + \frac{1}{2}(\alpha + \delta + 1), \qquad \lambda'' = p - \frac{1}{2}(\alpha + \delta + 1).$$

In like manner putting

$$\mu' = q + \frac{1}{2}(\beta + \delta + 1), \qquad \mu'' = q - \frac{1}{2}(\beta + \delta + 1);$$

$$\nu' = r + \frac{1}{2}(\gamma + \delta + 1), \qquad \nu'' = r - \frac{1}{2}(\gamma + \delta + 1),$$

we can show that

$$\begin{aligned} P_0^{\lambda'} &= H_{ad}, & P_0^{\lambda''} &= H_{bc}; \\ P_\infty^{\mu'} &= H_{bd}, & P_\infty^{\mu''} &= H_{ac}; \\ P_1^{\nu'} &= H_{cd}, & P_1^{\nu''} &= H_{ab}. \end{aligned}$$

Thus, H defines a binary shear of functions having the characteristic properties of the shear P with the indices just given,—for the functions defined by H behave regularly except in the points $0, \infty, 1$,— and we conclude:—*

$$\int_i^k z_1^{\alpha} \cdot z_2^{\beta} \cdot (z_1 - z_2)^{\gamma} \left[z_2 - \frac{z_1}{\lambda} \right]^{\delta} (zdz)$$

$$= \lambda^{-p + \frac{1}{2}(\alpha + 1 - \delta)} \cdot (1 - \lambda)^{-r + \frac{1}{2}(\gamma + \delta + 1)} \cdot P \left\{ \begin{matrix} 0 & \infty & 1 & \\ p + \frac{1}{2}(\alpha + \delta + 1) & q + \frac{1}{2}(\beta + \delta + 1) & r + \frac{1}{2}(\gamma + \delta + 1) & \lambda \\ p - \frac{1}{2}(\alpha + \delta + 1) & q - \frac{1}{2}(\beta + \delta + 1) & r - \frac{1}{2}(\gamma + \delta + 1) & \end{matrix} \right\}$$

$$= P \left\{ \begin{matrix} 0 & \infty & 1 & \\ \alpha + 1 & \beta + \delta + 1 & \gamma + \delta + 1 & \lambda \\ -\delta & 0 & 0 & \end{matrix} \right\}, \qquad (15)$$

by (12).

---

* See § 3, Art. 1, of this introduction.

Hence the integral in (4) or that in (3) furnishes a branch of a Riemann P-function, two of whose exponents are zero.

If in (6) we place

$$l = \alpha + 1\,, \qquad m = -\delta\,, \qquad n = \alpha + \gamma + 2\,, \qquad \lambda = \frac{1}{\lambda}\,,$$

and remember that $B(\alpha + 1, \gamma + 1)$ is a numerical factor, we will have

$$\Phi_{0i} = C.F(l, m, n, \lambda) = P_0^{\lambda''} \left\{ \begin{matrix} 0 & \infty & 1 & \\ 1 - n & l & n - l - m & \lambda \\ 0 & m & 0 & \end{matrix} \right\}, \tag{16}$$

where C is a constant different from zero.

Thus, in (15) and (16), we see how the Riemann P-function is related to the hypergeometric integral and function, and hence to the hypergeometric differential equation.

§ 4. *The Function* η.

5. Let the values of a complex variable z be pictured in the usual manner by the points of a plane—the "z-plane"—and the corresponding values of a function w of this variable by the points of a second plane—the "w-plane." The points of the w-plane may be called the *representations* of their corresponding points in the z-plane.

When the z-point is made to trace out any path in its plane, the corresponding w-point will trace out a path in its plane which we will call the representation of the path traced by the z-point.

If the w be an analytical function of z, the angle made by two curves in the z-plane will in general equal that made by their representations in the w-plane.* The representations may therefore be said to be *conformal*.

Any given simply connected region of the w-plane can be represented upon the positive or negative half of the z-plane, so that any three given points of the contour of this region will have corresponding to them any three points, arranged in the same sense, in the boundary of the half z-plane; i. e. in the real axis of the z-plane (Riemann).

6. If we take as our region in the w-plane a triangle bounded by the arcs of circles (such a triangle we shall hereafter call a circular triangle) with the angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ whose vertices L, M, N correspond respectively to the points a, b, c in the real axis of the z-plane, the representation is conformal everywhere except at the points a, b, c; i. e. the angles $\lambda\pi$, $\mu\pi$, $\nu\pi$ are represented in the z-plane by angles of the magnitude π, whereas all other angles in the w-region remain unchanged (Fig. 1).

* Forsyth's Theory of Functions, p. 11.

By means of this representation to every point of the positive half of the z-plane there will correspond a definite point w within the circular triangle.

This representation we take as the *definition* of a function η expressed more fully by the symbol $\eta\begin{bmatrix} a & b & c & z \\ \lambda & \mu & \nu & \end{bmatrix}$.

7.* It is a well known fact in the elementary theory of a complex variable, that, if we have before us two planes η and z, in which are two circular triangles, LMN and lmn respectively, whose angles arranged in a definite sense are $\lambda\pi$, $\mu\pi$, $\nu\pi$, the two triangles will be connected analytically by a linear relation

$$\eta = \frac{\alpha z + \beta}{\gamma z + \delta},$$

in which α, β, γ, δ are constants which may be complex, and

$$z = x + iy\,, \qquad \eta = \eta_1 + i\eta_2\,.$$

Either triangle may be bounded by straight lines.

If the two planes are connected by the linear relation

$$\bar{\eta} = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta},$$

where

$$\bar{\eta} = \eta_1 - i\eta_2 \quad \text{and} \quad \bar{z} = x - iy\,,$$

then the triangle $L'M'N'$ which corresponds to the triangle lmn of the z-plane will be a *reflection* (*spiegelung*) of LMN of the previous case. If the angles of LMN had the arrangement $\lambda\pi$, $\mu\pi$, $\nu\pi$ those of $L'M'N'$ will have the arrangement $\lambda\pi$, $\nu\pi$, $\mu\pi$ (see Fig. 2).

Therefore the symbol

$$\eta\begin{bmatrix} a & b & c & z \\ \lambda & \mu & \nu & \end{bmatrix}$$

represents not a unique, perfectly determined function but an entire shear of functions which arise out of a single one η_0 by substitutions of the form

$$\eta(z) = \frac{\alpha\eta_0 + \beta}{\alpha\gamma_0 + \delta}, \quad \text{or} \quad \eta(\bar{z}) = \frac{\alpha\eta_0 + \beta}{\gamma\eta_0 + \delta}.$$

8. From the definition of η it at once follows that we may develop it in the region surrounding an arbitrary point $z = z_0$ of the positive half of the z-plane in the form

$$\eta - \eta_0 = (z - z_0)\,.\,p\,(z - z_0)\,,$$

* Forsyth's Theory of Functions, p. 549.

Fig. 1.

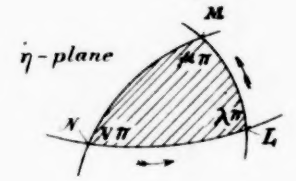
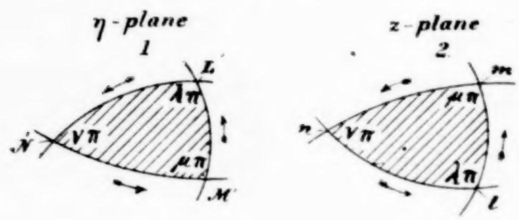


Fig. 2.



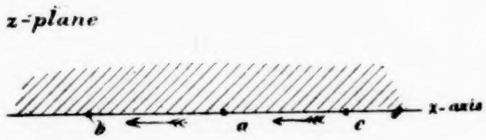


Fig. 4.

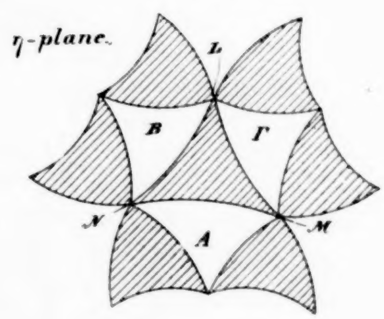


Fig. 3.

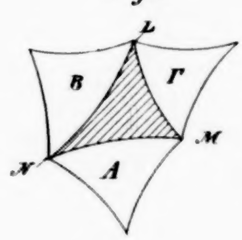


Fig. 5.

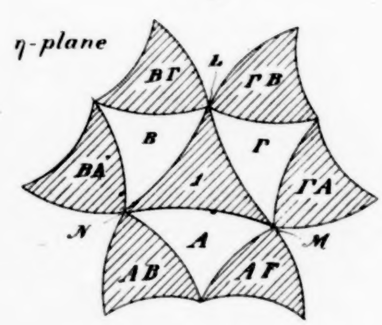
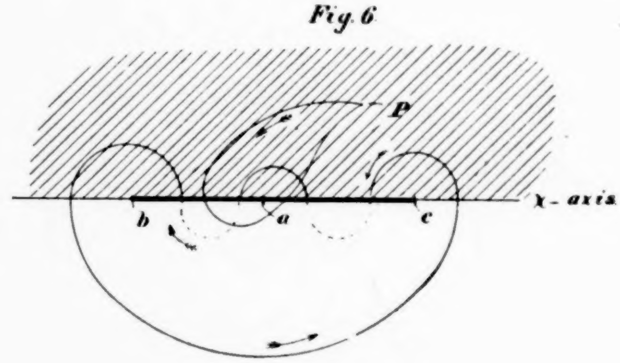


Fig. 6.



where p is a series arranged in ascending positive integral powers of $z - z_0$ whose first term does not vanish; except for the points $z = a, b, c$, where the developments have the form*

$$\begin{aligned} \eta - L &= (z-a)^{\lambda} . p(z-a), \\ \eta - M &= (z-b)^{\mu} . p(z-b), \qquad (17) \\ \eta - N &= (z-c)^{\nu} . p(z-c). \end{aligned}$$

Should z_0 or η_0 be infinite one must put in these formulæ instead of $z - z_0$, $\eta - \eta_0$ respectively, the values $\frac{1}{z}, \frac{1}{\eta}$. It is evident that $z = a, b, c$ are branch points of η.

9. Since a, b, c are branch points, the value of the η-function corresponding to any point in the negative half of the z-plane will be different according to the path taken by z in traveling from an internal point in the positive half of the z-plane to this point.

Let $z = z_0, z_1$ be two points in the positive and negative half of the z-plane, respectively, and η_0 and η_1 the corresponding values of the η-function. If η_0 be a point within the triangle LMN, then, as z travels from z_0 to z_1, say over the segment ab, η will travel from η_0 across LM to a point η_1 without the triangle LMN; similarly, if z in traveling from z_0 to z_1 should cross the segments bc, ca, respectively, then η would travel from η_0 across the sides MN, NL, respectively.

What representations then of the negative half of the z-plane do we obtain if we begin with the representation of the positive half of the z-plane and allow z to cross over bc, or ca, or ab into the negative half of the z-plane? What new representations of the positive half of the z-plane will arise if z passes back by various paths into the positive half of the z-plane? An answer to these questions is to be found by aid of the *principle of symmetry* due to Schwarz. We construct about our original triangle (always shaded) three symmetrical triangles which we shall designate by A, B, Γ (Fig. 3).

The triangles A, B, Γ are exactly those representations which η produces when z crosses (from the positive side of the real axis) over bc, ca, ab, respectively, into the negative half of the z-plane, for the sides of the η-triangle LM, MN, NL along which the new triangles arrange themselves correspond respectively to the segments ab, bc, ca.

We construct also the triangles symmetrical to A, B, Γ, which we shade as seen in Fig. 4, and which are representations of the *positive* half of the z-plane.

* H. Durège's *Elemente der Theorie der Functionen*, chap. vi.

Thus, in accordance with the principle of symmetry, we construct in the η-plane an unlimited number of triangles shaded alternately.

Every shaded triangle is a representation of the positive side of the z-plane. The blank triangles are representations of the negative half of the z-plane.

The function $\eta(z)$, i. e. the aggregate of all the branches of $\eta(z)$ which are evolved by analytical development from the original branch, thus becomes a multiform function of z, indeed, in the general case, a function of an infinite number of values.

We designate the primitive triangle by 1, and its three reflections by A, B, Γ. We shall designate, in general, each triangle according to the manner in which it is evolved, by reflections, from 1. Hence, the triangles in Fig. 4 should be marked as in Fig. 5. Every shaded triangle receives a designation which contains an even number of symbols; every triangle not shaded receives a designation which contains an odd number of symbols. All shaded triangles are therefore connected with the primitive triangle by direct conformal representation, and all blank triangles by indirect conformal representation; i. e. by substitutions which have, respectively, the forms

$$\eta' = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}, \qquad \eta'' = \frac{\alpha\bar{\eta} + \beta}{\gamma\bar{\eta} + \delta}.$$

Whence the multiform function $\eta(z)$ has the properties, that all the distinct branches $\eta_\kappa(z)$ which belong to the representation of the positive half of the z-plane are associated with $\eta_0(z)$ by linear substitutions of the form

$$\eta(z) = \frac{\alpha_\kappa\eta_0 + \beta_\kappa}{\gamma_\kappa\eta_0 + \delta_\kappa}, \tag{18}$$

while the distinct branches of $\eta_\kappa(\bar{z})$ which belong to the negative half of the z-plane are furnished by subjecting the primitive branch $\eta_0(z)$ to substitutions of the form*

$$\bar{\eta}_\kappa(\bar{z}) = \frac{\bar{\alpha}_\kappa\eta_0 + \bar{\beta}^\kappa}{\bar{\gamma}_\kappa\eta_0 + \bar{\delta}_\kappa}. \tag{19}$$

The coefficients α, β, γ, δ are determined by the fact that the original figure is given.

All substitutions, (18) and (19), arise out of combinations of those three substitutions which produce the reflections A, B, Γ, so that these substitutions form a group.

* The symbols $\bar{\eta}$ and $\bar{z}$ correspond to the variable z.

If, in Fig. 5, η travels from any point in triangle 1 to the corresponding point in any other triangle (shaded) ΓB, we must have each time, in the z-plane, a path encircling correspondingly the branch points a, b, c (in this case encircling a), and ending finally in the point of departure;* and conversely. If, therefore, we let A, B, C represent, respectively, the linear substitutions which produce the reflections A, B, Γ, we may say that to a positive encircling of a, or b, or c there will correspond in the η-plane a substitution A, B, C, and consequently, if we combine these three substitutions in the order in which z encircles a, b, and c, the function η will return to its original value (or $CBA = 1$).

That is, if z encircles a, then b, and finally c, the branch η_0, with which we begin will pass successively into $A(\eta_0)$, $BA(\eta_0)$, $CBA(\eta_0) = \eta_0$ (see Fig. 6).

10. The branches of the η-function obtained by analytical development from η_0 are all included in the system represented by $\dfrac{\alpha\eta_0 + \beta}{\gamma\eta_0 + \delta}$ when α, β, γ, δ are arbitrary, but by no means exhaust this system. It is this last which constitutes the general η-function.

Riemann shows that the substitutions A, B, C are determined essentially by the values of λ, μ, ν; that is, are so far determined as is possible if the choice of the particular branch η_0 is left arbitrary.

11. Since $\eta\begin{bmatrix} a & b & c & \\ \lambda & \mu & \nu & z \end{bmatrix}$ represents a shear of branch-functions all of which may be derived from a given primitive branch η_0 by the linear substitution

$$\eta = \frac{\alpha\eta_0 + \beta}{\gamma\eta_0 + \delta},$$

we may construct the differential equation which all branches of η will satisfy by forming the three differential coefficients η', η'', η''', and then eliminating between them and η the three ratios $\alpha : \beta : \gamma : \delta$. Such a process leads to the differential expression†

$$\frac{\eta'''}{\eta''} - \frac{3}{2}\left[\frac{\eta''}{\eta'}\right]^2 = [\eta]_z, \qquad (20)$$

which depends only on a, b, c, λ, μ, ν, and z.

12. Since all the branches of $\eta(z)$ arise out of a single primitive branch η_0 through linear substitutions of the form

$$\frac{\alpha_\kappa\eta_0 + \beta_\kappa}{\gamma\,\eta_0 + \delta_\kappa},$$

* In Fig. 6, see loop running from P about a.

† See Forsyth's Differential Equations, end of § 61.

and $[\eta]_z$ does not change if we substitute for η any linear function of η, $[\eta]_z$ must be a single-valued function of z. Further, our initial η_0 has been determined 1° by the selection we have made of the three points a, b, c, 2° by the three angles of the η-triangle $\lambda\pi$, $\mu\pi$, $\nu\pi$, and 3° by the particular position which we have assigned to the triangle in its plane. If we had chosen this position in some other manner, there might have appeared instead of η_0 any $\dfrac{\alpha\eta_0+\beta}{\gamma\eta_0+\delta}$ (where α, β, γ, δ are arbitrary). But this choice of position does not affect $[\eta]_z$.* Hence, the value of $[\eta]_z$ does not depend upon the position of the primitive η-triangle, but only upon z, a, b, c, λ, μ, ν, and moreover has simply a one-valued dependence upon these quantities. Accordingly, we write

$$[\eta]_z = \text{a one-valued function of } (a, b, c, \lambda, \mu, \nu; z). \tag{21}$$

13. We seek next the character of the function $[\eta]_z$. To accomplish this we study its behavior in the region about all possible positions of z_0 in the z-plane. To avoid diffuseness we suppose a, b, c finite. Then we have three kinds of points to distinguish,

$$\left.\begin{array}{l} z_0 \text{ finite and distinct from } a, b, c\,; \\ z_0 \text{ coincident with } a, \text{ or } b, \text{ or } c\,; \\ z_0 \text{ infinite.} \end{array}\right\} \tag{22}$$

Since $[\eta]_z$ remains unaltered if we put instead of η any $\dfrac{\alpha\eta+\beta}{\gamma\eta+\delta}$, it will be no essential restriction if we assume in the following calculation that η_0 corresponding to z_0 is finite. Hence we shall have respectively, in the three cases (22), for the region about $z = z_0$, the following developments:

$$\left.\begin{array}{l} \eta - \eta_0 = \alpha(z-z_0) + \beta(z-z_0)^2 + \ldots, \\ \eta - \eta_0 = (z-z_0)^\lambda \cdot [\alpha + \beta(z-z_0) + \ldots]\text{, or, etc.} \\ \eta - \eta_0 = \alpha z^{-1} + \beta z^{-2} + \gamma z^{-3} + \ldots. \end{array}\right\} \tag{24}$$

where α is supposed to be $\gtrless 0$. Substituting, successively, the values in (24) in $[\eta]_z$, we obtain the following values of this "parameter" in the regions about $z = z_0$, $z = a$, (b or c), $z = \infty$, respectively:

$$[\eta]_z = \frac{6(\alpha\gamma-\beta^2)}{\alpha^2} + \text{terms in ascending powers of } z - z_0,$$

$$[\eta]_z = \frac{\frac{1}{2}(1-\lambda^2)}{(z-a)^2} + \frac{(\lambda^2-1)\beta}{\lambda\alpha}\cdot\frac{1}{z-a} + \text{terms in ascending integral powers of } z - a,$$

$$[\eta]_z = \frac{6(\alpha\gamma-\beta^2)}{\alpha^2 z^4} + \text{terms in ascending integral powers of } z.$$

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* Forsyth's Differential Equations, § 62, Ex. 2.

It is a fundamental theorem* of the function theory, that a one-valued function of z which for the entire z-plane possesses the character of an integral function, and has only a finite number of points for which the function becomes infinite, and then to a finite degree, is a rational function of z; and, in fact, it is a simple matter to construct a rational function of z which for $z = a, b, c$ becomes infinite, and for $z = \infty$ vanishes to the fourth degree, as we know $[\eta]_z$ does, from the equations immediately preceding. This function, which is fully determined by these requirements, is

$$R(z) = \frac{1}{(z-a)(z-b)(z-c)} \left[\frac{\frac{1-\lambda^2}{2}(a-b)(a-c)}{x-a} + \frac{\frac{1-\mu^2}{2}(b-c)(b-a)}{x-b} + \frac{\frac{1-\nu^2}{2}(c-a)(c-b)}{x-c} \right], \tag{25}$$

and the differential equation sought is

$$[\eta]_z = R(z).$$

Thus, we see that, from the definition of η by means of conformal representation, we have ascended to the differential equation of the third order of which η is the solution.

We notice further that, if the points a, b, c should lie respectively at 0, ∞, 1, the discussion on p. 156 would lead to the particular case of (25)

$$[\eta]_z = \frac{1-\lambda^2}{2x^2} + \frac{1-\nu^2}{2(x-1)^2} + \frac{\lambda^2-\mu^2+\nu^2-1}{2x(x-1)}. \tag{26}$$

14. We next consider the question, whether η can be so separated that, when z makes a closed circuit in its plane, and $\eta = y_1/y_2$ experiences the substitution

$$\eta = \frac{\alpha\eta + \beta}{\gamma\eta + \delta},$$

y_1 and y_2 may experience integral *binary* linear substitutions of the form

$$y_1 = \alpha y_1 + \beta y_2, \qquad y_2 = \gamma y_1 + \delta y_2.$$

There are two ways of looking for an answer to this question; viz. by regarding y_1 and y_2 as functions of z, or as binary quantics in z_1, z_2 ($z = z_1/z_2$). The second, of course, coincides with the first when we suppose the degree of each quantic zero.

* Forsyth's Theory of Functions, p 71.

One such separation as we are seeking is given by the equations

$$y_1 = \frac{\eta}{\sqrt{\eta'}}, \qquad y_2 = \frac{1}{\sqrt{\eta'}}.$$

For, the derivative of $\dfrac{\alpha\eta + \beta}{\gamma\eta + \delta}$ being $\dfrac{(\alpha\delta - \beta\gamma)\eta'}{(\gamma\eta + \delta)^2}$, it is obvious that when η is transformed into $\dfrac{\alpha\eta + \beta}{\gamma\eta + \delta}$, y_1 and y_2 as just defined are transformed into $\dfrac{\alpha y_1 + \beta y_2}{\sqrt{\alpha\delta - \beta\gamma}}$ and $\dfrac{\gamma y_1 + \delta y_2}{\sqrt{\alpha\delta - \beta\gamma}}$, respectively, and therefore experience a binary transformation. The determinant of this transformation is especially simple, being equal to unity.

These values of y_1 and y_2 are, however, not the only ones which satisfy the given conditions. More generally, we may write

$$\left.\begin{aligned} Y_1 &= \frac{\eta}{\sqrt{\eta'}}(z-a)^\rho (z-b)^\sigma (z-c)^\tau . F(z)\,, \\ Y_2 &= \frac{1}{\sqrt{\eta'}}(z-a)^\rho (z-b)^\sigma (z-c)^\tau . F(z)\,; \end{aligned}\right\} \tag{27}$$

where ρ, σ, τ may represent any numbers whatever, and $F(z)$ a uniform function of z, or more briefly

$$Y_1 = \frac{\eta}{\sqrt{\eta'}} . \varphi(z)\,, \qquad Y_2 = \frac{1}{\sqrt{\eta'}} . \varphi(z)\,,$$

Y_1 and Y_2 will experience, as well as y_1 and y_2, a binary linear substitution when z encircles a, or b, or c.

If, now, we put $y = c_1 y_1 + c_2 y_2$, then construct the first and second differential coefficients of y and between them and y eliminate c_1 and c_2, we will obtain the differential equation of the second order which y satisfies. It is

$$y'' + \tfrac{1}{2} R(z)\, y = 0\,.^* \tag{28}$$

where $[\eta]_z = R(z)$ is the equation of which η is the solution.

Moreover since

$$y_1 = \frac{Y_1}{\varphi(z)}, \qquad y_2 = \frac{Y_2}{\varphi(z)}$$

the differential equation which Y_1 and Y_2 satisfy is the linear equation of the second order,

$$Y'' + pY' + qY = 0\,, \tag{29}$$

* Compare Forsyth's Differential Equations, § 61.

where

$$p = -\frac{2\varphi'}{\varphi}, \qquad q = \frac{P_2}{2} + 2\frac{\varphi'^2}{\varphi^2} - \frac{\varphi''}{\varphi}.$$

Thus, starting with η defined by the representations which it produces, we have been led to the general linear differential equation of the second order,—the inverse of the ordinary method of procedure. Our point of view is that of Riemann.

15. We show first now that the functions y_1 and y_2 are not the simplest of those satisfying the given conditions as might be supposed. For, as we shall show, $z = \infty$, in addition to $z = a, b, c$, is a singular point for y_1 and y_2. To prove this, we put instead of z, z_1/z_2, and at the same time suppose η to be split into two forms of the nth degree in z_1, z_2; i. e.

$$\eta = \frac{H_1(z_1, z_2)}{H_2(z_1, z_2)},$$

where n is arbitrary.

If, for the moment, we put

$$Z_1 = H_1(z_1, 1), \qquad Z_2 = H_2(z_1, 1), \qquad \eta = \frac{z_1}{z_2};$$

we have

$$y_1 = \frac{Z_1}{\sqrt{Z_2 Z_1' - Z_1 Z_2'}}, \qquad y_2 = \frac{Z_2}{\sqrt{Z_2 Z_1' - Z_1 Z_2'}};$$

or

$$y_1 = \frac{H_1}{\sqrt{z_2\left[H_2\dfrac{\partial H_1}{\partial z_1} - H_1\dfrac{\partial H_2}{\partial z_1}\right]}}, \qquad y_2 = \frac{H_2}{\sqrt{z_2\left[H_2\dfrac{\partial H_1}{\partial z_1} - H_1\dfrac{\partial H_2}{\partial z_1}\right]}}. \tag{30}$$

But, according to Euler's theorem,

$$nH_1 = z_1\frac{\partial H_1}{\partial z_1} + z_2\frac{\partial H_1}{\partial z_2}, \qquad nH_2 = z_1\frac{\partial H_2}{\partial z_1} + z_2\frac{\partial H_2}{\partial z_2};$$

hence

$$y_1 = \frac{\sqrt{n}\,.\,H_1}{\sqrt{z_2^2\left[\dfrac{\partial H_1}{\partial z_1}\dfrac{\partial H_2}{\partial z_2} - \dfrac{\partial H_2}{\partial z_1}\dfrac{\partial H_1}{\partial z_2}\right]}}, \quad y_2 = \frac{\sqrt{n}\,.\,H_2}{\sqrt{z_2^2\left[\dfrac{\partial H_1}{\partial z_1}\dfrac{\partial H_2}{\partial z_2} - \dfrac{\partial H_2}{\partial z_1}\dfrac{\partial H_1}{\partial z_2}\right]}};$$

or, if we represent the functional determinant of H_1 and H_2 by (H_1, H_2),

$$y_1 = \frac{\sqrt{n}}{z_2}\cdot\frac{H_1}{\sqrt{(H_1, H_2)}}, \qquad y_2 = \frac{\sqrt{n}}{z_2}\cdot\frac{H_2}{\sqrt{(H_1, H_2)}}; \tag{31}$$

which shows that y_1 and y_2 have the singular point $z = z_1/0 = \infty$.

16. We therefore proceed to select such branches Y_1 and Y_2, from (27), as have singular points $z = a, b, c$ only. To this end we choose an $F(z)$ which has no singular points; i. e. set it equal to a constant C. We have only to find the condition which ρ, σ, τ fulfil in order that $z = \infty$ may not be a singular point. We set $a = a_1/a_2$, $b = b_1/b_2$, $c = c_1/c_2$; whence from (27) and (31) we have

$$Y_1 = \frac{\sqrt{n} \, . \, H_1}{\sqrt{(H_1, H_2)}} \cdot \frac{(za)^\rho (zb)^\sigma (zc)^\tau}{z_2^{\rho+\sigma+\tau+1} a_2^\rho b_2^\sigma c_2^\tau} \cdot C, \text{ etc}.$$

We put $C = a_2^\rho b_2^\sigma c_2^\tau / \sqrt{n}$, thus we have

$$Y_1 = \frac{H_1}{\sqrt{(H_1, H_2)}} \cdot \frac{(za)^\rho (zb)^\sigma (zc)^\tau}{z_2^{\rho+\sigma+\tau+1}},$$

$$Y_2 = \frac{H_2}{\sqrt{(H_1, H_2)}} \cdot \frac{(za)^\rho (zb)^\sigma (zc)^\tau}{z_2^{\rho+\sigma+\tau+1}}. \tag{32}$$

If these expressions are not to be infinite for $z = \infty$, then $z_2^{\rho+\sigma+\tau+1}$ must be equal to 1. This will happen when $\rho + \sigma + \tau = -1$, which is therefore the condition sought.

In this way we are brought at once to the Riemann P-function. For, putting $Y = c_1 Y_1 + c_2 Y_2$, we have a binary shear, or group of branch functions, before us which possess all the properties by means of which the Riemann P-function was defined (see § 3).

Because it is evident that Y has no singular points excepting $z = a, b, c$, and that as z makes circuits about a, b, c it will pass into such new branch functions only as belong to the shear $Y = c_1 Y_1 + c_2 Y_2$ itself.

From (27) if we return to non-homogeneous variables and set $F(z) = l$ as in (32), we may deduce

$$Y = \frac{c_1 \eta + c_2}{\sqrt{\eta'}} (z - a)^\rho (z - b)^\sigma (z - c)^\tau. \tag{33}$$

If any branch η becomes η_a for $z = a$, then

$$Y = \frac{\eta - \eta_a}{\sqrt{\eta'}} (z - a)^\rho (z - b)^\sigma (z - c)^\tau, \tag{34}$$

a branch function included in (33), which may be developed in the following manner:—

We obtained earlier for $\eta - \eta_a$ the development

$$\eta - \eta_a = (z - a)^\lambda \, p(z - a);$$

accordingly, η' has the form

$$(z-a)^{\lambda-1}\,.\,p_1(z-a)\,;$$

hence our branch function will have the form

$$(z-a)^{\frac{1}{2}(\lambda+1)+\rho}\,.\,p_2(z-a)\,. \tag{35}$$

But the shear (33) contains also the branch function

$$\frac{\sqrt{\eta'}}{c_2}(z-a)^{\rho}(z-b)^{\sigma}(z-c)^{\tau}\,.$$

By a similar calculation this branch has, in the region about $z=a$, the development

$$(z-a)^{\frac{-\lambda+1}{2}+\rho}\,.\,p_3(z-a)\,.$$

We proceed in a similar manner for the developments of branches of the shear (33) in the regions about $z=b$, $z=c$, and find in (27) or in (33) functions with the exponents

$$\tfrac{1}{2}(\pm\lambda+1)+\rho\,,\qquad \tfrac{1}{2}(\pm\mu+1)+\sigma\,,\qquad \tfrac{1}{2}(\pm\nu+1)+\tau\,,$$

for the points

$$z=a\,,\qquad z=b\,,\qquad z=c\,,$$

respectively.

Thus, in the shear are two functions for each of the singular points a, b, c, with determinate exponents belonging to them. Moreover, since

$$\rho+\sigma+\tau=-1\,,$$

the sum of these six exponents is equal to $+1$; which is the same as the condition

$$\lambda'+\lambda''+\mu'+\mu''+\nu'+\nu''=1$$

in the definition of the Riemann P-function.*

The shear (27) or (33) is, therefore, the Riemann P-function

$$P\left|\begin{array}{cccc} a & b & c & \\ \rho+\dfrac{1}{2}(1+\lambda) & \sigma+\dfrac{1}{2}(1+\mu) & \tau+\dfrac{1}{2}(1+\nu) & z \\ \rho+\dfrac{1}{2}(1-\lambda) & \sigma+\dfrac{1}{2}(1-\mu) & \tau+\dfrac{1}{2}(1-\nu) & \end{array}\right|, \tag{37}$$

which is evidently the most general P-function belonging to $\eta\begin{bmatrix} a & b & c & z \\ \lambda & \mu & \nu & \end{bmatrix}$.

* See (10) and what follows.

Having set out with our η-function we have thus been led to the Riemann P-function.

But our method of attack leads us at once to a generalization of this function. The condition

$$\rho + \sigma + \tau = -1$$

has nothing to do with the definition of η, viz. $\eta = y_1/y_2$. It is simply the condition that Y_1, Y_2 are of the degree zero in z_1, z_2, i. e. functions of z. The ρ, σ, τ in (27) may be taken as arbitrary quantities. Y_1, Y_2 then appear as quantics of the degree $\rho + \sigma + \tau + 1$.

Out of the generalized Y_1, Y_2 we construct the binary shear $c_1 Y_1 + c_2 Y_2$, which Klein in order to recall the Riemann P-function represents by the symbol H. We can define this shear of functions by means of its properties in a manner exactly similar to that employed by Riemann with his P. Klein represents by H the scheme,

$$H \left| \begin{array}{cccc} a_1, a_2 & b_1, b_2 & c_1, c_2 & \\ \rho + \dfrac{1}{2}(1+\lambda) & \sigma + \dfrac{1}{2}(1+\mu) & \tau + \dfrac{1}{2}(1+\nu) & z_1, z_2 \\ \rho + \dfrac{1}{2}(1-\lambda) & \sigma + \dfrac{1}{2}(1-\mu) & \tau + \dfrac{1}{2}(1-\nu) & \end{array} \right|, \tag{38}$$

which differs from the scheme employed by Riemann only in this that the sum of the exponents need no longer be equal to unity. If we make the sum of the exponents equal to unity, which is always admissible, then H reduces to the Riemann P-function.

We wish to consider particularly the special case, called by Klein that of the normal H of the second species.

If λ, μ, ν be assumed positive, and we put

$$\rho = \tfrac{1}{2}(\lambda - 1), \quad \sigma = \tfrac{1}{2}(\mu - 1), \quad \tau = \tfrac{1}{2}(\nu - 1),$$

we have

$$Y_1 = \frac{H_1}{\sqrt{(H_1, H_2)}} \cdot (za)^{\frac{1}{2}(\lambda-1)} (zb)^{\frac{1}{2}(\mu-1)} (zc)^{\frac{1}{2}(\nu-1)}$$

$$Y_2 = \frac{H_2}{\sqrt{(H_1, H_2)}} \cdot (za)^{\frac{1}{2}(\lambda-1)} (zb)^{\frac{1}{2}(\mu-1)} (zc)^{\frac{1}{2}(\nu-1)}. \tag{39}$$

Thus we obtain as our binary shear

$$H \left| \begin{array}{cccc} a & b & c & \\ \lambda & \mu & \nu & z \\ 0 & 0 & 0 & \end{array} \right|, \tag{40}$$

which has the degree $\frac{1}{2}(\lambda + \mu + \nu - 1)$ in z_1, z_2. We can carry through an analogous discussion when the signs of one or more of the quantities λ, μ, ν are changed, and obtain $\frac{1}{2}(\pm \lambda \pm \mu \pm \nu - 1)$ as the degrees of the *eight* corresponding *shears*.

I.

This paper has for its object the study of the conformal representations produced by the η-function (as defined in the introduction) when Klein's *normal* Π of the second species (see (40)) contains a single algebraic function, or what amounts to the same thing, when the *hypergeometric differential equation* which Π satisfies has a *single algebraic solution*.

17. In the first section of his memoir (Crelle's Journal, Bd. 75, § 1) Schwarz has shown that, if the differential equation,

$$\frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \cdot \frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)}\,y = 0\,, \tag{41}$$

of which the Gaussian hypergeometric series $F(\alpha, \beta, \gamma\,;\, x)$* is a particular solution or integral, has a single algebraic solution, this solution must have the form

$$y_1 = x^a (1-x)^c\, g(x)\,. \tag{42}$$

In the case of this equation the singular points are $0, \infty, 1$. By the same reasoning, if the differential equation of which

$$\Pi \begin{vmatrix} a & b & c & \\ \lambda & \mu & \nu & z \\ 0 & 0 & 0 & \end{vmatrix}$$

is the solution—which differs from that considered by Schwarz only in that the singular points a, b, c are left arbitrary—have a single algebraic solution, that solution must be of the form

$$\Pi_1 = (za)^{\xi\lambda}(zb)^{\xi'\mu}(zc)^{\xi''\nu}\,\varphi_k(z_1, z_2)\,, \tag{43}$$

when written homogeneously; in which φ_k is a rational integral function of the degree k in z_1, z_2, while ξ, ξ', ξ'' may be 0 or 1.

There are four cases of the expression in (43) to be considered ;† viz.

$$\left.\begin{aligned} &\alpha) \quad \Pi_1 = \varphi_k(z_1, z_2)\,, \\ &\beta) \quad \Pi_1 = (za)^\lambda \cdot \varphi_k(z_1, z_2)\,, \\ &\gamma) \quad \Pi_1 = (zb)^\mu \cdot (zc)^\nu \cdot \varphi_k(z_1, z_2)\,, \\ &\delta) \quad \Pi_1 = (za)^\lambda \cdot (zb)^\mu \cdot (zc)^\nu \cdot \varphi_k(z_1, z_2)\,. \end{aligned}\right\} \tag{44}$$

* Forsyth's Differential Equations, § 113 et seq.

† The order in which λ, μ, ν are selected is arbitrary.

Since the degree of H was found from (40) to be $\frac{1}{2}(\lambda+\mu+\nu-1)$, k has in these four cases the values

$$\frac{1}{2}(\lambda+\mu+\nu-1),\quad \frac{1}{2}(-\lambda+\mu+\nu-1),\quad \frac{1}{2}(\lambda-\mu-\nu-1),$$
$$\frac{1}{2}(-\lambda-\mu-\nu-1). \tag{45}$$

Whence, since k must be an integral number $\geqq 0$, the case δ) cannot exist.

18. In the introduction we have defined a branch of H_1, the normal H of the second species, by the formula

$$H_1 = \frac{z_2}{\sqrt{\eta'}} \cdot (za)^{\frac{1}{2}(\lambda-1)} \cdot (zb)^{\frac{1}{2}(\mu-1)} \cdot (zc)^{\frac{1}{2}(\nu-1)}, \tag{46}$$

where

$$\eta' = \frac{d\eta}{dz} = z_2^2 \frac{d\eta}{(zdz)}.$$

Hence, solving (46) for η' and substituting successively for H_1 the values given in α), β), γ) of (44), we obtain

$$\left.\begin{aligned}
&\alpha) \qquad \eta = \int \frac{(za)^{\lambda-1}(zb)^{\mu-1}(zc)^{\nu-1}(zdz)}{\varphi_k^2(z_1, z_2)}, \\
&\beta) \qquad \eta = \int \frac{(za)^{-\lambda-1} \cdot (zb)^{\mu-1}(zc)^{\nu-1}(zdz)}{\varphi_k^2(z_1, z_2)}, \\
&\gamma) \qquad \eta = \int \frac{(za)^{\lambda-1}(zb)^{-\mu-1}(zc)^{-\nu-1}(zdz)}{\varphi_k^2(z_1, z_2)};
\end{aligned}\right\} \tag{47}$$

i. e., in these cases, the function of η has been expressed by means of a simple indefinite integral.

We remark that, aside from the points of discontinuity which $\varphi_k(z_1, z_2) = 0$ furnishes, the integral α) is everywhere finite, but the integral β) is infinite at $z = a$, and γ) at $z = b$, $z = c$. Since $\varphi_k(z_1, z_2)$ satisfies a linear homogeneous differential equation, its roots are all distinct and are different from a, b, c. Each of the integrals in (47) has, therefore, for $\varphi_k = 0$, k simple algebraic points of discontinuity.

19. If z encircles a, the integral η in α), of (47), becomes

$$\eta' = e^{2\pi i\lambda} \int \frac{(za)^{\lambda-1}(zb)^{\mu-1}(zc)^{\nu-1}(zdz)}{\varphi_k^2};$$

and the most general transformation which η experiences when z takes a course which involves circuits about a, b, c is of the form

$$\eta' = \alpha\eta + \beta; \tag{48}$$

The same remarks apply to η in cases β) and γ).

If, in (48), $\eta = \infty$, then $\eta' = \infty$. Whence we have to deal with linear substitutions for which the point $\eta = \infty$ remains fixed.

And if, instead of the branch of η defined by (47), we take another branch which arises from this by any linear substitution $\frac{\alpha\eta + \beta}{\gamma\eta + \delta}$, we will have to deal with such a group of linear substitutions as leave fixed a definite point of our new η'-plane.

II.

Conformal Representations produced by the η-function when λ, μ, ν are Real.

20. The equation $\pm\lambda \pm \mu \pm \nu = 2k + 1$ can be satisfied not only for real but also for imaginary λ, μ, ν.

When λ, μ, ν are real we cut the z-plane along the *real axis*, then study the representations of the two half planes. If we multiply a single triangle of the η-plane by reflection, in accordance with the principle of symmetry, there will arise in the manner previously explained the corresponding group of η-functions. As we have just seen, for the substitutions which transform the functions of this group into one another, a particular point of the η-plane will remain fixed. Hence the three sides of each of the η-triangles pass through this point. If the fixed point mentioned be moved to $\eta = \infty$, the η-triangle becomes rectilinear, and our special case (λ, μ, ν real) is thus characterized by the fact that our circular η-triangle may assume the form of a *rectilinear triangle.*

21. If λ, μ, ν are complex, but none of them pure imaginaries

$$(\lambda = \lambda' + i\lambda'', \qquad \mu = \mu' + i\mu'', \qquad \nu = \nu' + i\nu''),$$

instead of the old system of cuts along the real axis conceive three cuts made in the z-plane from any auxiliary point O in the plane to the points a, b, c, respectively, and approaching these points assymptotically (spirally).* When z is made to cross one of these cuts any particular branch η_0 of the η-function is transformed into another branch of this function, connected with η_0—as all branches of η are—by a linear substitution. The corresponding configuration in the η-plane will therefore be a hexagon—when the cuts in the z-plane are properly chosen, a circular hexagon (Fig. 7). The angles of the points a, b, c of the figure are $2\lambda'\pi$, $2\mu'\pi$, $2\nu'\pi$, respectively, and the angles at the intermediate points, which are all representations of the point O, are equal to the angles between the corresponding lines Oa, Ob, Oc in the z-plane.

In the particular case with which we are concerned—in which one branch of the P-function corresponding to our η-function is algebraic—the substitu-

* In order to avoid the occurrence of such spirals in the corresponding η-figure. In the immediate neighborhood of $z = 0$, for instance, $\eta = z^\lambda = (\rho e^{i\theta})^{\lambda' + i\lambda''} = \rho^{\lambda'} e^{-\theta\lambda''} \cdot e^{i(\theta\lambda' + \lambda'' \log\rho)}$.

tions by which the sides meeting in a, in b, in c are connected with each other are all of the form $\eta' = a\eta + \beta$; a substitution which, interpreted, geometrically, represents those transformations of a plane into itself which arise by combining rotations (including translations) with a transformation of every figure of the plane into a similar figure.

From (47) we see that the representations in cases α), β), γ) differ from one another in this respect: in case α) the vertices of our triangle (or hexagon lie in the finite portion of the η-plane; but in β) one vertex, and in γ) two vertices, lie at infinity (see Fig. 8).

22. Since λ, μ, ν are not necessarily less than 1, but may have any magnitudes whatever, it is obvious that a triangle of which $\lambda\pi$, $\mu\pi$, $\nu\pi$ are the angles will not, in general, be a triangle in the elementary sense of the word, but may be any figure bounded by three arcs of circles, which may be less or greater than whole circumferences, and having angles of any magnitude whatever. The surface of such a triangle will consist of that of the elementary triangle determined by the three angular points, together with multiples of half-planes. As will be seen later on, the general triangle in any case may be constructed from the elementary triangle having the same vertices by the attachment of half-planes.

23. In case λ, μ, ν be real, $\varphi_k(z_1, z_2)$ is itself real, and it is natural to inquire: How many real roots has $\varphi_k = 0$ in the segments $b \frown c$, $c \frown a$, $a \frown b$ of the real axis? Also, how many imaginary roots has φ_k? If the numbers referred to be represented by l, m, n, $2p$, respectively, then

$$l + m + n + 2p = k = \tfrac{1}{2}(\lambda + \mu + \nu - 1). \tag{49}$$

It can be proven that l represents the number of times that the side of the η-triangle opposite the angle $\lambda\pi$ passes through the point at infinity; that m and n stand in the same relation to the other two sides; and that p represents the number of times the body of the triangle passes through the point at infinity.

If, then, the form of any rectilinear triangle be given, with the angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, the values of l, m, n, p are known directly.*

Case α): $\lambda + \mu + \nu = 2k + 1$.

24. In this case, in accordance with (45), $\lambda + \mu + \nu = 2k + 1$ is a positive odd integer.

In investigating the possible forms of the triangles corresponding to this

* These values have been determined by Stieltjes and Hilbert in their investigations by algebraic methods of hypergeometric series of a finite number of terms (Crelle, Bd. 103, 1887.)

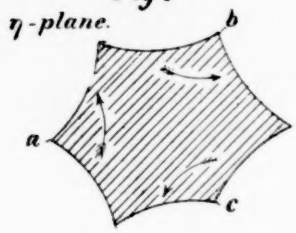


Fig 7

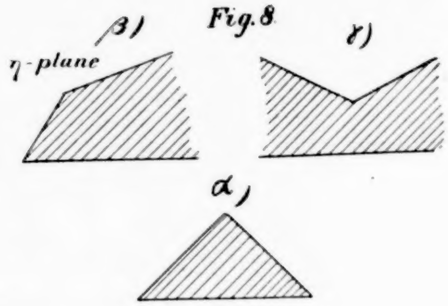


Fig. 8

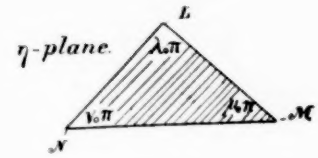


Fig. 9.

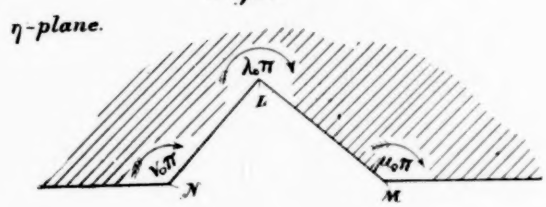


Fig. 10.

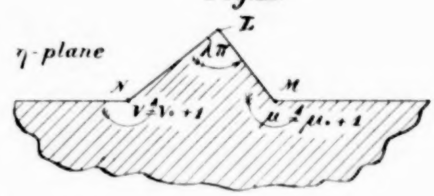


Fig. 11.

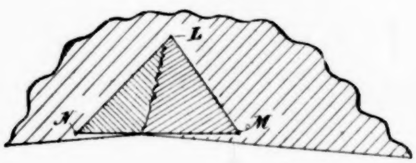


Fig. 12.

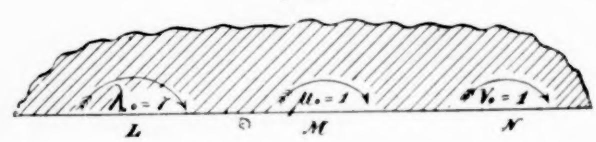


Fig. 14.

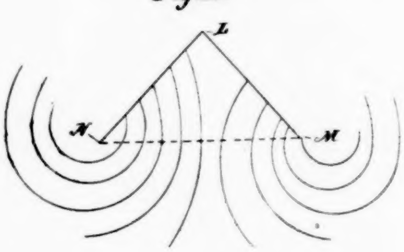


Fig. 13.

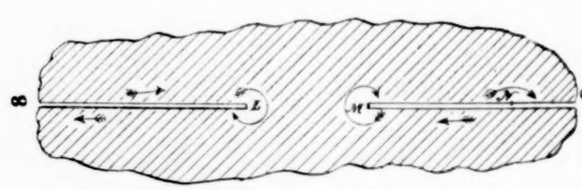


Fig. 15.

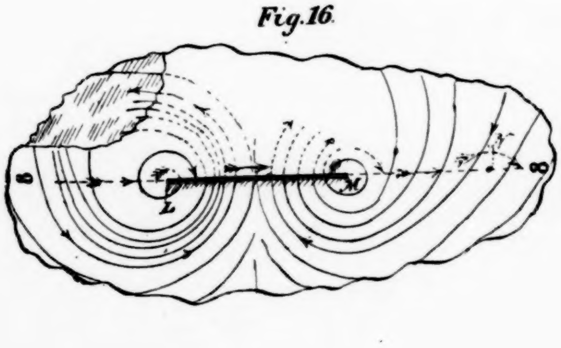


Fig. 16.

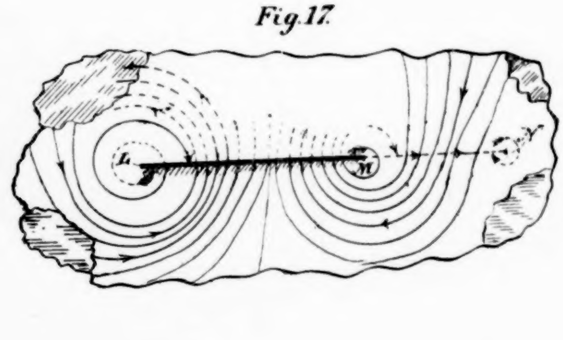


Fig. 17.

case, we begin with a process which we may characterize as that of *arithmetic reduction.*

Let

$$\left.\begin{aligned} \lambda &= \xi(\lambda) + (\lambda)\,, \\ \mu &= \xi(\mu) + (\mu)\,, \\ \nu &= \xi(\nu) + (\nu)\,, \end{aligned}\right\} \tag{50}$$

in which $\xi(\lambda)$, $\xi(\mu)$, $\xi(\nu)$ and (λ), (μ), (ν) are the *integral* and *fractional* parts of λ, μ, ν, respectively. Either or all of (λ), (μ), (ν) may be zero. If (λ), (μ), (ν) are all zero then λ, μ, ν are integers. Since $\lambda + \mu + \nu = 2k + 1$, we shall have three general cases to discuss, according as $(\lambda) + (\mu) + (\nu) = 0$, 1, or 2; i. e. according as $\xi(\lambda) + \xi(\mu) + \xi(\nu) =$ an odd integer, an even integer, or an odd integer.

We suppose $\xi(\lambda) \geqq \xi(\mu) \geqq \xi(\nu)$, which is not a restriction, and distinguish between the cases

$$\xi(\lambda) \leqq \xi(\mu) + \xi(\nu)\,,$$
$$\xi(\lambda) \geqq \xi(\mu) + \xi(\nu)\,.$$

Disregarding for the present the case $(\lambda) + (\mu) + (\nu) = 0$, we have four cases between which to distinguish. These I shall call A_{11}, A_{12}, A_{21}, A_{22}, as indicated in the following table:

	$(\lambda) + (\mu) + (\nu) = 1$	$(\lambda) + (\mu) + (\nu) = 2$
$\xi(\lambda) \leqq \xi(\mu) + \xi(\nu)$	A_{11}	A_{21}
$\xi(\lambda) \geqq \xi(\mu) + \xi(\nu)$	A_{12}	A_{22}

25. *Reduction of case* A_{11}.

Assume $\xi(\lambda) = b + c$, $\xi(\mu) = c + a$, $\xi(\nu) = a + b$, in which a, b, c are positive integral numbers, and take as the angles of a *reduced triangle*,* $\lambda_0\pi$, $\mu_0\pi$, $\nu_0\pi = (\lambda)\pi$, $(\mu)\pi$, $(\nu)\pi$, respectively. Throughout this discussion by a reduced triangle is meant what is left of the given triangle on removing as many half-planes as possible from it. Often in giving the meaning of an angle, say $\lambda\pi$, we shall omit writing the π. Hence by (50) we may ascend to the general triangle through the system of equations

$$\left.\begin{aligned} \lambda &= \lambda_0 + b + c\,, \\ \mu &= \mu_0 + c + a, \\ \nu &= \nu_0 + a + b\,. \end{aligned}\right\} \tag{51}$$

* Klein: Ueber die Nullstellen der hypergeometrischen Reihe, Math. Annalen, Vol. 37, p. 580.

26. *Reduction of case* A_{12}.

In a manner analogous to that employed in A_{11}, we put

$$\xi(\mu) = c\,, \quad \xi(\nu) = b\,, \quad \xi(\lambda) - \xi(\mu) - \xi(\nu) = 2A\,,$$

in which $2A$ is an even integer ; for the reduced-triangle

$$\lambda_0 = (\lambda)\,, \quad \mu_0 = (\mu)\,, \quad \nu_0 = (\nu)\,.$$

Finally by what precedes and by (50), we may ascend to the general triangle by means of the formulæ

$$\left.\begin{aligned} \lambda &= \lambda_0 + b + c + 2A\,, \\ \mu &= \mu_0 + c\,, \\ \nu &= \nu_0 + b\,. \end{aligned}\right\} \tag{52}$$

27. *Reduction of case* A_{21}.

In this case $\xi(\lambda) + \xi(\mu) + \xi(\nu)$ is odd ; and since $\xi(\lambda) \geqq \xi(\mu) + \xi(\nu)$, we shall define three numbers a, b, $c \geqq 0$, by means of the equations

$$\xi(\lambda) = b + c + 1\,, \quad \xi(\mu) = c + a\,, \quad \xi(\nu) = a + b\,.$$

In this case we put for our reduced triangle

$$\lambda_0 = (\lambda) + 1\,, \quad \mu_0 = (\mu)\,, \quad \nu_0 = (\nu)\,;$$

whence $\lambda_0 + \mu_0 + \nu_0 = 3$; then by (50) we ascend to the general triangle by means of the formulæ

$$\left.\begin{aligned} \lambda &= \lambda_0 + b + c\,, \\ \mu &= \mu_0 + c + a, \\ \nu &= \nu_0 + a + b\,. \end{aligned}\right\} \tag{53}$$

28. *Reduction of case* A_{22}.

Putting

$$\xi(\lambda) = b + c + 2A + 1\,, \quad \xi(\mu) = c\,, \quad \xi(\nu) = b\,;$$

$$\lambda_0 = (\lambda) + 1\,, \qquad \mu_0 = (\mu), \quad \nu_0 = (\nu)\,;$$

we have

$$\left.\begin{aligned} \lambda &= \lambda_0 + b + c + 2A\,, \\ \mu &= \mu_0 + c\,, \\ \nu &= \nu_0 + b\,. \end{aligned}\right\} \tag{54}$$

The formulæ for the reduced and general triangles in (51), (52), (53), (54) satisfy the general condition that $\lambda + \mu + \nu$ is an odd positive integer $= 2k + 1$, as they should.

29. *Geometric construction.* In cases A_{11}, A_{12}, we have for our reduced triangle $\lambda_0 + \mu_0 + \nu_0 = 1$. The reduced triangle is therefore the ordinary plane triangle of Euclid (Fig. 9), and each angle is less than π. On the other hand in cases A_{21}, A_{22} $\lambda_0 + \mu_0 + \nu_0 = 3$, λ_0 alone being greater than π. Hence our reduced triangle in these cases will have the form shown in Fig. 10. In the case of Fig. 9,

$$l_0 = 0\,, \quad m_0 = 0\,, \quad n_0 = 0\,, \quad p_0 = 0\,, \quad k_0 = 0$$

by (49); in that of Fig. 10,

$$l_0 = 1\,, \quad m_0 = 0\,, \quad n_0 = 0\,, \quad p_0 = 0\,, \quad k_0 = 1\,.$$

From the reduction formulæ (51)–(54) we see that these are certain geometrical operations to be applied to the several reduced triangles a times, b times, c times, and A times in order to develop from them the general triangles. These operations are of two kinds, the *lateral* and the *polar* attachment of half-planes In the first case we attach half-planes to the sides of the reduced triangle. If, for example, we attach a half-plane to the horizontal side of Fig. 9, our triangle becomes the triangle of Fig. 11. The horizontal side has in this way been transformed into its complement: μ_0, ν_0 have each been increased by unity, while l has become 1. A repetition of the operation will give rise to branch points at the two lower vertices, the surface of the triangle having been increased by a whole-plane. The horizontal side again becomes the side of the original triangle or l again becomes zero, but on the other hand, p now equals 1.

In the *polar* attachment we draw a line from the vertex to any point of the base and attach a half-plane along the line as a *branch-section* (Verzweigungsschnitt) (Fig. 12). The angle λ will then become $\lambda_0 + 2\pi$, and l will equal 1.

In cases A_{11}, A_{21}, the arithmetic reduction was made by the equations

$$\lambda = \lambda_0 + b + c\,, \quad \mu = \mu_0 + c + a\,, \quad \nu = \nu_0 + a + b\,;$$

in cases A_{12}, A_{22} by the equations

$$\lambda = \lambda_0 + b + c + 2A\,, \quad \mu = \mu_0 + c\,, \quad \nu = \nu_0 + b\,.$$

The geometric meaning of this is, that, in cases A_{11}, A_{21}, we pass from the reduced triangle to the general, by the lateral attachment of a, b, and c half-

planes, respectively, to the three sides of the reduced triangle; in cases A_{12}, A_{22}, by the lateral attachment of b and c half-planes, respectively, to two of the sides, and by the polar attachment of A half-planes to the angle included between these sides.

30. If the general triangle arises from the reduced by the lateral attachment of a, b, c half-planes, and we put

$$a = 2a' + \varepsilon_1\,, \quad b = 2b' + \varepsilon_2\,, \quad c = 2c' + \varepsilon_3\,,$$

where ε_i may $= 0$ or 1, then in accordance with (49)

$$l = \varepsilon_1\,, \quad m = \varepsilon_2\,, \quad n = \varepsilon_3\,, \quad p = a' + b' + c'\,.$$

On the other hand the effect of the polar attachment of A half-planes to the vertex L is to cause the side opposite L to pass through infinity A times.

From the results of the above discussion we construct the following table for the various values of l, m, n, and p, in the different cases under a):

	l	m	n	p
A_{11}	ε_1	ε_2	ε_3	$a' + b' + c'$
A_{21}	$1 - \varepsilon_1$	ε_2	ε_3	$a' + b' + c' + \varepsilon_1$
A_{12}	A	ε_2	ε_3	$b' + c'$
A_{22}	$1 + A$	ε_2	ε_3	$b' + c'$

31. *Case in which λ, μ, ν are integral and $\lambda + \mu + \nu$ is an odd positive integer* $= 2k + 1$.

To explain the geometric construction in case λ, μ, ν are integers, we have simply to put for our reduced triangle

$$\lambda_0 = 1\,, \quad \mu_0 = 1\,, \quad \nu_0 = 1\,,$$

and ascend to the general triangle by means of the equations

$$\lambda = \lambda_0 + b + c\,, \quad \mu = \mu_0 + c + a\,, \quad \nu = \nu_0 + a + b\,, \tag{55}$$

where, as above, a, b, c are supposed to be positive integers. Our reduced triangle will have the form given in Fig. 14.

1° In the special case, $\lambda = 2$, $\mu = 2$, $\nu = 1$, the η-triangle has the form given in Fig. 15.

2° For the case, $\lambda = 3$, $\mu = 3$, $\nu = 1$, see Fig. 16.

3° For the case, $\lambda = 4$, $\mu = 3$, $\nu = 2$, see Fig. 17.

The general triangle in (55) is completed by lateral attachment of half-planes.

We may also have polar-attachment of half-planes, the branch-section coinciding with the η-real-axis.

In analogy with the table at the end of Art. 30, we will have

$$\left.\begin{array}{lllll} A_{00}) & l = 1 + \varepsilon_1, & m = \varepsilon_2, & n = \varepsilon_3, & p = a' + b' + c'\,; \\ A_{10}) & l = A + 1, & m = \varepsilon_2, & n = \varepsilon_3, & p = b' + c'\,. \end{array}\right\} \qquad (56)$$

32. The cases in which one or two of the quantities λ, μ, ν are zero should be discussed under β) and γ). Because, if $\lambda = 0$ in α) of (47), then the vertex of the η-triangle corresponding to the point $z = 0$ would lie at infinity, and all such cases belong to case β). If, say, $\mu = 0 = \nu$ in α) of (47), then the vertices M, N of the η-triangle would lie at infinity ; such cases belong to γ).

The case $\lambda = \mu = \nu = 0$, can never exist ; because k is then a negative quantity $(= -\frac{1}{2})$, which is contrary to hypothesis.

$$\textit{Case } \beta): \quad -\lambda + \mu + \nu = 2k + 1\,.$$

33. In this case, in accordance with (45), $-\lambda + \mu + \nu = 2k + 1$* is a positive integer. Hence there always exists the inequality $\lambda < \mu + \nu$.

In case β), as in case α), we may employ the system of equations (50) subject to the conditions $-\lambda + \mu + \nu = 2k + 1$ and $\lambda > \mu + \nu$. There are now only two general cases to be considered,

$$-(\lambda) + (\mu) + (\nu) = 1\,, \qquad (\mathrm{b}_1)$$

$$-(\lambda) + (\mu) + (\nu) = 0\,; \qquad (\mathrm{b}_2)$$

whereas in case α) there were three.

Considered geometrically the negative sign before λ indicates that the vertex of the η-triangle which corresponds to the point $z = a$ of the z-plane, is at ∞, as is shown by equation (47), β).

(b_1). $-(\lambda) + (\mu) + (\nu) = 1$. This case may be readily explained, and the corresponding geometric operations accomplished, if we put

$$\xi(\lambda) = b + c\,, \quad \xi(\mu) = c + a\,, \quad \xi(\nu) = a + b\,;$$

$$\lambda_0 = (\lambda)\,, \qquad \mu_0 = (\mu)\,, \qquad \nu_0 = (\nu)\,;$$

whence

$$\lambda = \lambda_0 + b + c\,, \quad \mu = \mu_0 + c + a\,, \quad \nu = \nu_0 + a + b\,,$$

where, as before, a, b, c are positive integers.

* It is a matter of convention that λ is negative. Either λ, or μ, or ν may be negative.

Evidently, $-\lambda_0 + \mu_0 + \nu_0 = 1$ and $-\lambda + \mu + \nu = 2a + 1$, as they should be, in conformity with the conditions of our problem. The reduced triangle will take the form given in Fig. 18,* and the general triangle may be constructed from it by the lateral attachment of a, b, c half-planes. For the special cases, $\lambda = \lambda_0$, $\mu = \mu_0 + 1$, $\nu = \nu_0 + 1$; and $\lambda = \lambda_0 + 1$, $\mu = \mu_0 + 1$, $\nu = \nu_0 + 2$, see Fig. 19 and Fig. 20, respectively. Fig. 19 is constructed by the lateral attachment of a half-plane to the side LN in Fig. 19.

(b_2). $-(\lambda) + (\mu) + (\nu) = 0$. In this case we may have 1° $(\lambda) = (\mu) + (\nu)$, or 2° $(\lambda) = (\mu) = (\nu) = 0$.

1° $(\lambda) = (\mu) + (\nu)$. In order to construct all the triangles in this case, we have simply to put for our reduced triangle

$$\lambda_0 = (\lambda)\,, \qquad \mu_0 = (\mu) + 1\,, \qquad \nu_0 = (\nu)\,;$$

and to ascend to the general triangle we place

$$\xi(\lambda) = b + c\,, \qquad \xi(\mu) = c + a + 1\,, \qquad \xi(\nu) = a + b\,;$$

whence, finally,

$$\left.\begin{aligned} \lambda &= \lambda_0 + b + c\,, \\ \mu &= \mu_0 + c + a\,, \\ \nu &= \nu_0 + a + b\,; \end{aligned}\right\} \tag{58}$$

which satisfy the condition $-\lambda + \mu + \nu = 2k + 1$. The reduced triangle will have the form shown in Fig. 21. For, from the triangle MNQ,

$$\lambda_0 = \nu_0 + (\mu)\,; \text{ i. e. } (\lambda) = (\nu) + (\mu)\,.$$

For the special case $\lambda = \lambda_0$, $\mu = \mu_0 + 1$, $\nu = \nu_0 + 1$, see Fig. 22.

2° $(\lambda) = (\mu) = (\nu) = 0$. The discussion of this case is perfectly analogous to that of the corresponding case under a) in which λ, μ, ν are each integral, the only difference being that the vertex of the η-triangle which corresponds to $-\lambda$ in this case lies at infinity (see Fig. 23).

34. (b_3). If μ and ν be fractional and $\lambda = 0$, but as before $-\lambda + \mu + \nu = 2k + 1$, we put

$$\left.\begin{aligned} \lambda &= 0\,, \\ \mu &= \xi(\mu) + (\mu)\,, \\ \nu &= \xi(\nu) + (\nu)\,, \end{aligned}\right\} \tag{59}$$

* From the triangle $L'MN$, Fig. 18,

$$\lambda_0 + 1 - \mu_0 + 1 - \nu_0 = 1\,; \text{ i. e. } -\lambda_0 + \mu_0 + \nu_0 = 1\,.$$

Fig. 18.

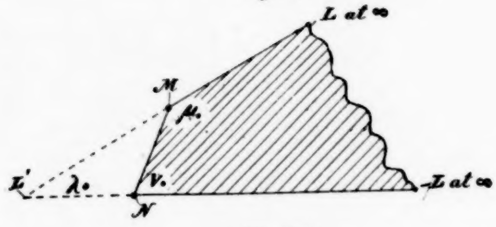


Fig. 19.

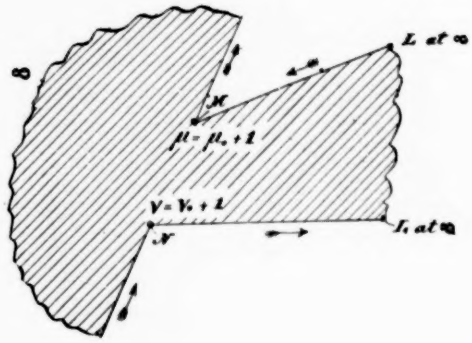


Fig. 20.

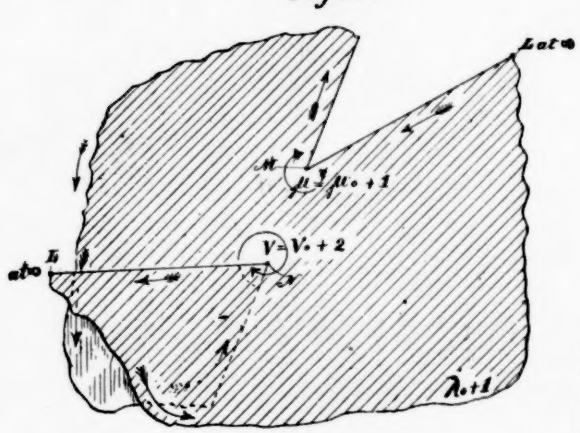


Fig. 21.

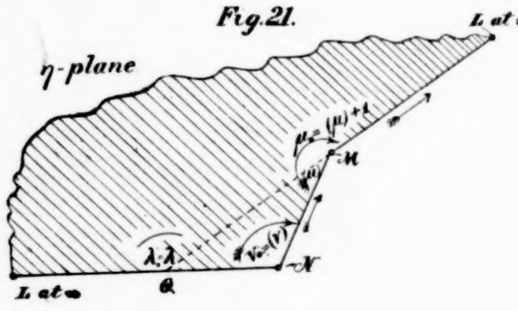


Fig. 22.

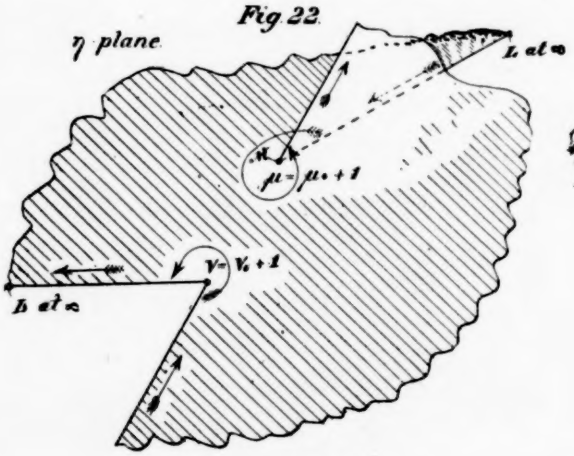


Fig. 23.

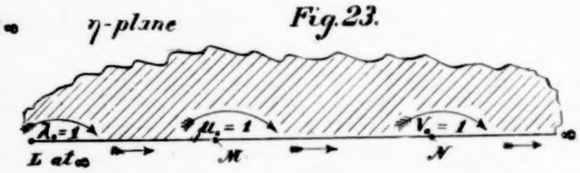


Fig. 24.

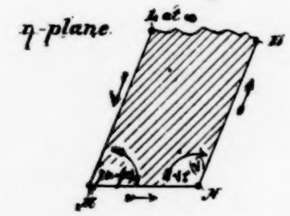
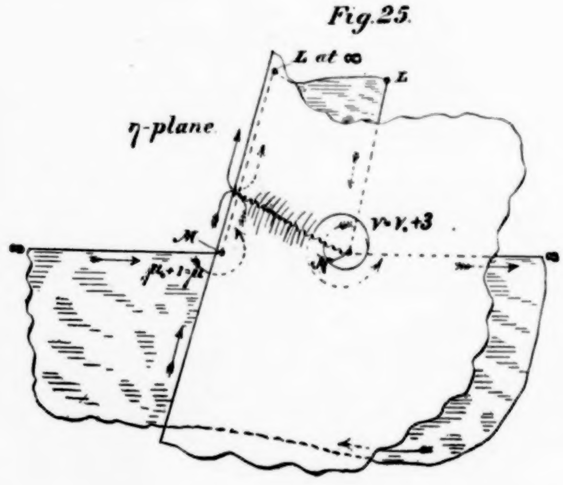


Fig. 25.



where $(\mu) + (\nu)$ may equal 0 or 1. If $(\mu) = (\nu) = 0$, then λ, μ, ν are integral, the case just discussed.

If $(\mu) + (\nu) = 1$, we put for the reduced triangle $\lambda_0 = 0$, $\mu_0 = (\mu)$, $\nu_0 = (\nu)$ (see Fig. 24), and ascend to the general triangle by putting $\xi(\lambda) = 0$, $\xi(\mu) = b$, $\xi(\nu) = b + 2A$; whence

$$\left.\begin{aligned} \lambda &= 0 , \\ \mu &= \mu_0 + b , \\ \nu &= \nu_0 + b + 2A , \end{aligned}\right\} \tag{60}$$

where, as above, b and A are positive integers. We can construct this general triangle by the polar attachment of A half-planes and the lateral attachment of b half-planes to the triangle in Fig. 24.

If, in particular, $b = 1$, $A = 1$ so that $\lambda = 0$, $\mu = \mu_0 + 1$, $\nu = \nu_0 + 3$, the corresponding triangle will have the form given in Fig. 25.

35. (b_4). If λ be integral, then $(\lambda) = 0$, and we shall have the same reduced triangle as that employed in (b_3). The only difference between this case and (b_3) is, that in case (b_3) in ascending to the general triangle we employed both lateral and polar attachment of half-planes, while to ascend to the general triangle in case (b_4) we have to employ only lateral attachment of half-planes to MN, or NL, or LM.

36. To find the number of times a side, or the surface of our triangles, passes through infinity, in cases (b_1), (b_2), (b_3), (b_4) we proceed as in Art. 31, (56).

From (49)

$$l + m + n + 2p = \tfrac{1}{2}(-\lambda + \mu + \nu - 1) = k ; \tag{61}$$

and, as before, we put

$$a = 2a' + \varepsilon_1 , \quad b = 2b' + \varepsilon_2 , \quad c = 2c' + \varepsilon_3 ,$$

in which ε_1, ε_2, $\varepsilon_3 = 0$ or 1. Substituting the formulæ for the general triangle, we will have for (b_1), (b_2), (b_4),

$$p = 0 , \quad l = 2a' + \varepsilon_1 , \quad m = n = 0 ;$$

and, for (b_3),

$$p = 0 , \quad l = 2b' + \varepsilon_2 , \quad m = 0 , \quad n = A .$$

Here p represents the number of times the body of the η-triangle covers the point or infinity, and l, m, n, the number of times the sides of the triangles opposite the vertices L, M, N, respectively, pass through infinity.

It is to be noticed, in case β), that the point L is a branch point.

$$Case\ \gamma): \lambda - \mu - \nu = 2k + 1 .$$

37. In this case $\lambda - \mu - \nu = 2k + 1$ is a positive integer. As in a) we employ the set of equations (50). There are two general cases,

$$(\lambda) - (\mu) - (\nu) = -1 , \tag{c$_1$}$$

$$(\lambda) - (\mu) - (\nu) = 0 . \tag{c$_2$}$$

(c_1). $(\lambda) - (\mu) - (\nu) = -1$. In this case we take for the reduced triangle

$$\lambda_0 = (\lambda) + 2 , \quad \mu_0 = (\mu) , \quad \nu_0 = (\nu) .$$

The form of this triangle will be that given in Fig. 26, the angles M and N lying at ∞.

The general triangle may be constructed by putting

$$\xi(\lambda) = b + c + 2(A + 1) , \quad \xi(\mu) = c , \quad \xi(\nu) = b ,$$

whence

$$\lambda = \lambda_0 + b + c + 2A , \quad \mu = \mu_0 + c , \quad \nu = \nu_0 + b . \tag{62}$$

We effect this construction by the lateral attachment of b and c half-planes and the polar attachment of A half-planes.

For the special case,

$$\lambda = \lambda_0 + 1 , \quad \mu = \mu_0 , \quad \nu = \nu_0 + 1 ,$$

we have a triangle of the form given in Fig. 27.

The case, $\lambda = \lambda_0 + 3$, $\mu = \mu_0$, $\nu = \nu_0 + 1$, i. e. when $A = 1$, $c = 0$, $b = 1$, is constructed by the diagonal polar attachment of a half-plane, in Fig. 27, to a branch section running from the vertex L to the opposite side.

(c_2). 1° $(\lambda) = (\mu) + (\nu)$; $(\lambda), (\mu), (\nu)$ not $= 0$. We may here put for the reduced triangle

$$\lambda_0 = (\lambda) + 1 , \quad \mu_0 = (\mu) , \quad \nu_0 = (\nu) .$$

For, from the triangle $LM'N'$, Fig. 28,

$$\nu_0 + \mu_0 + 2 - \lambda_0 = 1 ; \text{ i. e. } -(\lambda) + (\mu) + (\nu) = 0 .$$

For the general triangle we may put

$$\xi(\lambda) = b + c + 2A + 1 , \quad \xi(\mu) = c , \quad \xi(\nu) = b ;$$

whence

$$\lambda = \lambda_0 + b + c + 2A , \quad \mu = \mu_0 + c , \quad \nu = \nu_0 + b , \tag{63}$$

The general triangle may, therefore, be constructed from Fig. 28 by the lateral attachment of b and c half-planes and the polar attachment of A half-planes.

The triangle for the special case, $\lambda = \lambda_0 + 2$, $\mu = \mu_0$, $\nu = \nu_0$, is given in Fig. 29.

2° $(\lambda) = (\mu) + (\nu)$, $(\lambda) = (\mu) = (\nu) = 0$. In this case λ, μ, ν are integral numbers. We take for the reduced triangle, $\lambda_0 = 1$, $\mu_0 = 0$, $\nu_0 = 0$; i. e. the triangle given in Fig. 30.

Since $\lambda - \mu - \nu$ is an odd integer, the general triangle may be constructed by means of the formulæ

$$\lambda = \lambda_0 + b + c + 2A\,, \quad \mu = \mu_0 + c\,, \quad \nu = \nu_0 + b\,. \tag{64}$$

In this case we shall have the lateral attachment of b and c half-planes and the polar attachment of A half-planes.

In Figs. 31 and 32, the forms of the triangles are given for the special cases

$$\lambda = \lambda_0 + 1\,, \quad \mu = \mu_0 + 1\,, \quad \nu = \nu_0 = 0\,;$$

$$\lambda = \lambda_0 + 3\,, \quad \mu = \mu_0 + 1\,, \quad \nu = \nu_0 = 0\,,$$

respectively.

38. (c_3). If $\mu = 0 = \nu$, $\lambda = 2k + 1$ is an odd integer. To construct all triangles belonging to this case we take the reduced triangle employed in 37, 2°. The general triangle may be constructed by the polar attachment of k half-planes along the branch-cut running from L to the opposite side (see Fig. 33).

39. (c_4). $\lambda - \mu - \nu = 2k + 1$, λ only being integral.

We put

$$\lambda = \xi(\lambda) + 0\,, \quad \mu = \xi(\mu) + (\mu)\,, \quad \nu = \xi(\nu) + (\nu)\,;$$

whence

$$\lambda - \mu - \nu = 2k + 1 = \xi(\lambda) - \xi(\mu) - \xi(\nu) - (\mu) - (\nu)\,.$$

Hence we may have either

$$-(\mu) - (\nu) = 0\,, \tag{1}$$

or

$$-(\mu) - (\nu) = -1\,. \tag{2}$$

Case (1), $-(\mu) - (\nu) = 0$, may be satisfied, and thus only, by putting $(\mu) = (\nu) = 0$. λ, μ, ν would then be integral numbers. This case has already been discussed in Art. 38.

(2). If $-(\mu) - (\nu) = -1$, we put for the reduced triangle

$$\lambda_0 = 2\,, \quad \mu_0 = (\mu)\,, \quad \nu = (\nu)\,;$$

then

$$\lambda_0 - \mu_0 - \nu_0 = 2 - (\mu) - (\nu) = 1\,,$$

as it should be (see Fig. 34). We construct the general triangle by means of

the system of equations

$$\lambda = \lambda_0 + b + c + 2A \,, \quad \mu = \mu_0 + c \,, \quad \nu = \nu_0 + b \,,$$

by the lateral attachment of b and c half-planes and the polar attachment of A half-planes.

The form of the triangle in the special case, $\lambda = 3$, $\mu = (\mu)$, $\nu = (\nu) + 1$, may be seen in Fig. 35.

40. In regard to the number of times the sides and surface of the η-triangles pass through infinity, it may be remarked that by substitution in the formula

$$l + m + n + 2p = \tfrac{1}{2}(\lambda - \mu - \nu - 1) = k \,, \tag{49}$$

we obtain

$$l = A \,, \quad m = n = p = 0 \,,$$

in all the four cases $(c_1) - (c_4)$.

Hence the side of the η-triangle opposite the vertex L passes A times through infinity, the two sides opposite M and N and the body of the η-triangle do not pass through the point at ∞.

III.

Conformal Representations produced by the η-function when λ, μ, ν are complex.

Case a): $\lambda + \mu + \nu = 2k + 1$.

41. Suppose the z-plane be cut along three lines extending from a, b, c to an auxiliary point O. The corresponding representation in the η-plane is a hexagon, three of whose vertices, 1, 3, 5, correspond to O, and the remaining three, 2, 4, 6, to a, b, c, respectively. The sides meeting in each of the last three points are connected with one another by a linear substitution. In order to form a clear conception of this representation, let us first consider in connection with it the case in which λ, μ, ν are real. It is assumed throughout that $\lambda + \mu + \nu = 2k + 1$.

42. We begin with the reduced case, $\lambda_0 + \mu_0 + \nu_0 = 1$, and draw the z-plane, as before, cut along the *real* axis, and construct the triangle in the η-plane which corresponds to the positive half of the z-plane (see Fig. 36). About this triangle of the η-plane we construct three representations of the negative half of the z-plane, each of these representations corresponding to one of the three segments bc, ca, ab over which we may cross when, starting in the positive half of the z-plane, we pass into the negative half of the same. An auxiliary point O is then taken in one of these side triangles and joined by straight lines to the vertices of the same, thus dividing the triangle into three

Fig. 26.

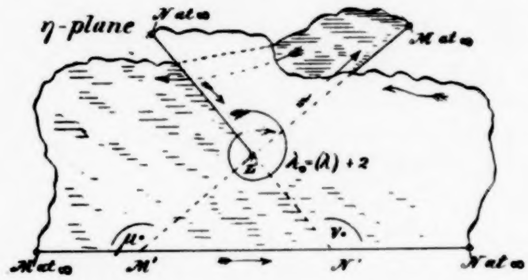


Fig. 27.

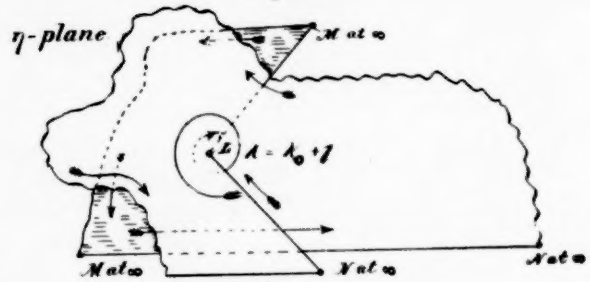


Fig. 28

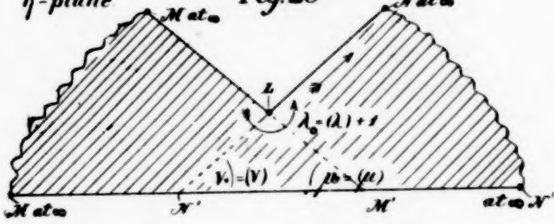


Fig. 30.

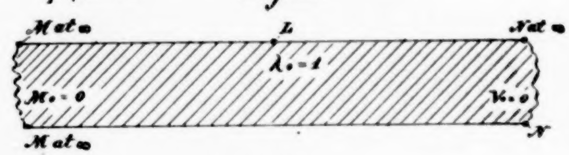


Fig. 29.

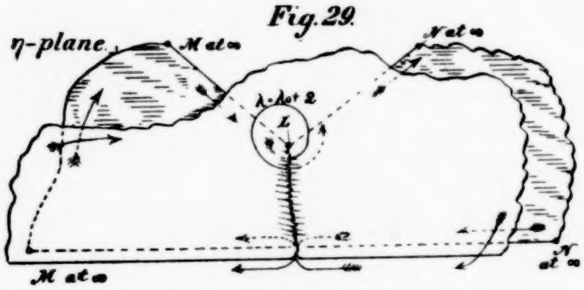


Fig. 31.

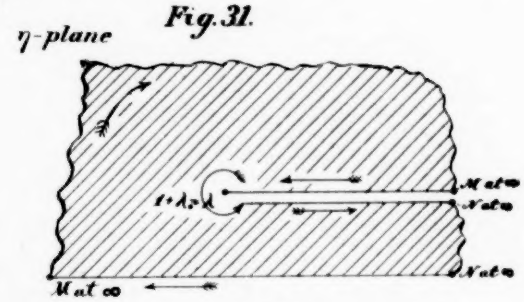


Fig. 32.

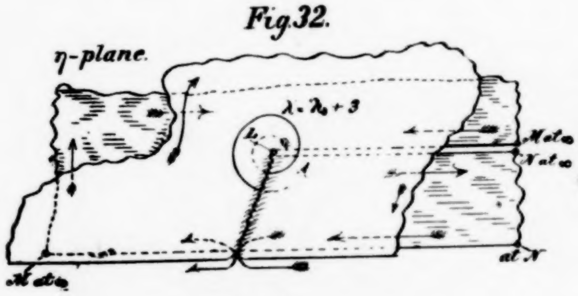


Fig. 33.

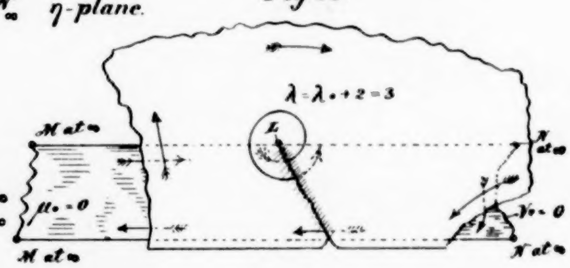


Fig. 34.

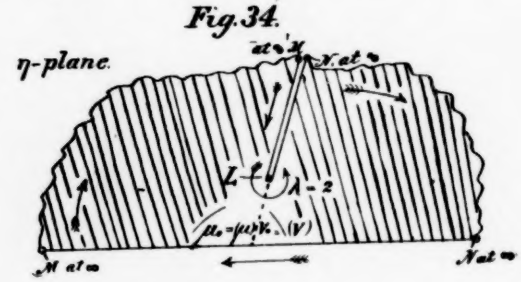
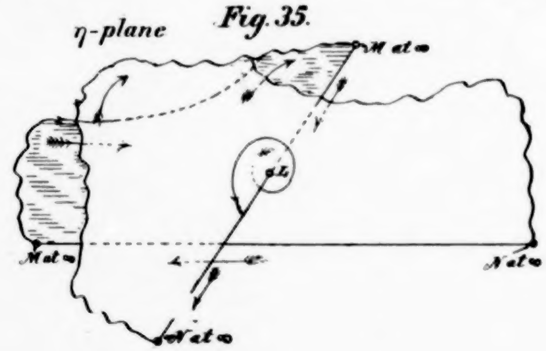


Fig. 35.



portions, which we shall call 1, 2, 3 (see Fig. 37), and a corresponding construction is made in each of the remaining side triangles. We next form the corresponding figure in the z-plane and in this figure number 1, 2, 3 the regions which correspond to those similarly numbered in Fig. 37 (see Fig. 38).

If now instead of cutting the z-plane along the axis of real numbers, we cut it in the lines Ob, Oc, Oa evidently the corresponding projection in the η-plane will be the hexagon which is ruled in Fig. 37, and which we shall once more draw so as to indicate the correspondence of its path to the positive and negative half of the z-plane (see Fig. 39). Its sides as indicated in the figure are connected by rotations of the η-plane about the vertices a, b, c, the amplitude of these rotations being $2\lambda\pi$, $2\mu\pi$, $2\nu\pi$, respectively.

We consider next the special case of the general kind of representation which arises when the point O coincides with the point a (see Fig. 40). Here the regions 2 and 3 vanish, and the representation degenerates into the quadrilateral, i. e. is simply the representation of the z-plane when cut from b through a to c in a straight line (see Fig. 40). This would be the projection in the reduced case, $\lambda_0 + \mu_0 + \nu_0 = 1$.

43. The next question that arises is, how can one pass from the reduced case to the general case λ, μ, ν? Or, in other words, what in our new representation takes the place of *lateral* and *polar* attachment of half-planes? It is of course the attachment of whole planes. In fact there is no difficulty presented in attaching a whole plane along one of the segments MN, NL, LM (in Figs. 39 and 40), the segment in case it does not lie exactly on the border of the figure to be regarded as a branch which unites the two branch points which lie in the vertices. Also, the *polar* attachment of whole-planes is easy to understand, if (confining ourselves to the point L) we hold fast to Fig. 40. In this case one has simply to draw a diagonal from L to L, the dotted line in Fig. 40, and to this dotted diagonal as a branch-section attach whole-planes.

In Fig. 39 we draw from L two dotted lines to two sides of the hexagon which belong together and attach a half-plane along each (see Fig. 41).

The same method of procedure may be applied to the case in which the reduced triangle is defined by $\lambda_0 + \mu_0 + \nu_0 = 3$.

We have, then, the triangle shown in Fig. 42, and develop from it, by means of a single reflection, the quadrilateral of Fig. 43, whose surface passes once through infinity. Here we can just as readily attach whole-planes as in the case just considered; only, for example, the dotted diagonal corresponding to that drawn in Fig. 40 connecting LL must now pass through infinity.

44. Having finished this preparatory discussion, we now proceed to the consideration of complex exponents,

$$\lambda = \lambda' + i\lambda'', \quad \mu = \mu' + i\mu'', \quad \nu = \nu' + i\nu''.$$

We suppose all that has been done with the real exponents, in the preceding section, to be done with the *real parts* of the complex exponents.

We shall, for the present, exclude the cases in which λ', μ', ν' are integral or zero, and put

$$\lambda' = \xi(\lambda') + (\lambda'), \quad \mu' = \xi(\mu') + (\mu'), \quad \nu = \xi(\nu') + (\nu'),$$

and distinguish two cases, $(\lambda') + (\mu') + (\nu') = 1$ or 2. In the first case we put

$$\lambda'_0 = (\lambda'), \quad \mu'_0 = (\mu'), \quad \nu'_0 = (\nu'),$$

and in the second,

$$\lambda'_0 = (\lambda') + 1, \quad \mu'_0 = (\mu'), \quad \nu'_0 = (\nu'),$$

and we assume

$$\xi(\lambda') \geqq \xi'(\mu) \geqq \xi(\nu').$$

Our problem now is to develop a representation of the z-plane for the reduced cases,

$$\lambda_0 = \lambda'_0 + i\lambda'', \quad \mu_0 = \mu'_0 + i\mu'', \quad \nu_0 = \nu'_0 + i\nu''.$$

From it we shall easily pass by attachment of whole planes to the cases of arbitrary values of λ, μ, ν.

45. We begin with the case in which λ, μ, ν are complex and $\lambda_0 + \mu_0 + \nu_0 = 1$. We construct in the η-plane a quadrilateral whose sides are straight lines (Fig. 44, which is a generalization of that in Art. 40, the subdivision into symmetric triangles disappearing). The angles at M and N are respectively $2\mu'_0\pi$, $2\nu'_0\pi$, and the two sides radiating from M, as indicated in Fig. 44, are not of the same length, but have the ratio $s : se^{-2\mu''\pi}$. Similarly, the two sides radiating from N have the ratio $t : te^{-2\nu''\pi}$. In fact these two pairs of sides are connected by the loxodromic substitutions,*

$$\eta' - M = e^{2i(\mu'_0 + i\mu'')\pi}(\eta - b),$$

$$\eta' - N = e^{2i(\nu'_0 + i\nu'')\pi}(\eta - c),$$

respectively; i. e. by rotations of $2\mu'_0\pi$ and $2\nu'_0\pi$, associated with extensions in the ratios $e^{-2\mu''\pi}$ and $e^{-2\nu''\pi}$.

The quadrilateral, constructed in this manner, which results if the z-plane be cut spirally† from M and N through L, is the representation we are seeking.

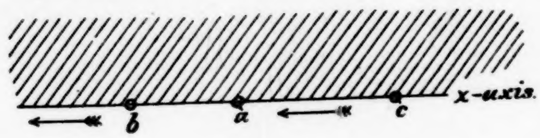
46. PROOF. In accordance with § 4 of the Introduction it follows at once, from the construction, that the quadrilateral may be regarded as a complete

* Lithographic notes of Klein's lectures on differential equations, winter semester, 1890–91.

† Klein's lectures on differential equations, 1890–1891, p. 161.

Fig. 36.

z-plane.



η-plane.

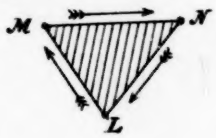


Fig. 37.

η-plane.

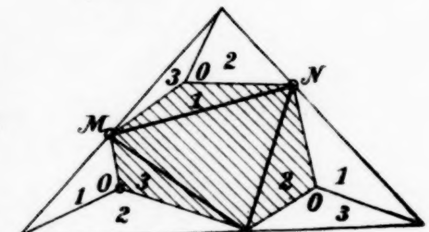


Fig. 38.

z-plane.

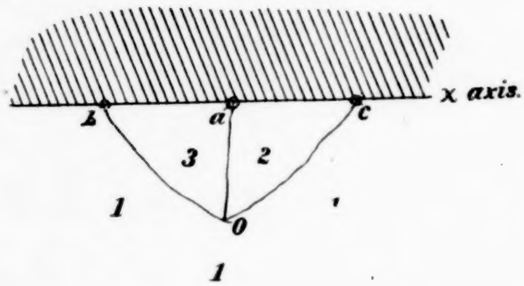
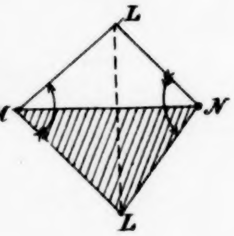


Fig. 40.

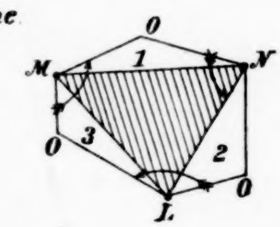
η-plane



z-plane.

Fig. 39.

η-plane



representation of the entire z-plane, and as such, defines a function $\eta(a, b, c, x)$; also, the function $\eta(a, b, c, x)$ so defined has for the points $z = b, c$ the exponents μ_0, ν_0. It only remains to show that for the point $z = a$ the exponent is $\lambda_0 = \lambda'_0 + i\lambda''_0$. By hypothesis, $\lambda_0 + \mu_0 + \nu_0 = 1$; whence $\lambda'' + \mu'' + \nu'' = 0$, and $\lambda'_0 + \mu'_0 + \nu'_0 = 1$. Hence it must be shown that for $z = a$ the exponent is $\lambda_0 = 1 - \mu'_0 - \nu'_0 - i(\mu'' + \nu'')$; or, in other words, it must be shown that the two sides radiating from L, in Fig. 44, are connected by a loxodromic substitution which has an angle of rotation equal to $2(1 - \mu'_0 - \nu'_0)\pi$ and causes an extension of the plane in the ratio $e^{2(\mu''+\nu'')\pi}$.

To show this, we annex to our quadrilateral by means of the loxodromic substitution employed in Fig. 44, a second quadrilateral (see Fig. 45). We add the two dotted lines MN and NM so forming a second quadrilateral $NMLMN$. This second quadrilateral constitutes a new representation of the z-plane, obtained by cutting the z-plane along a line extending from a through b to c. Since the sum of the angles of a quadrilateral equals 2π, and by construction the angle MLM (Fig. 45) is equal to the sum of the angles at the vertices L above and L below in the original quadrilateral (Fig. 44), the angle MLM will equal $2(1 - \mu'_0 - \nu'_0)\pi$. Also, the two sides LM to the left and LM to the right, in Fig. 45, have lengths whose ratio is $s : se^{-2(\mu''+\nu'')\pi}$, which supplies the proper modulus $e^{-2(\mu''+\nu'')\pi}$.

Hence, as was to be shown, the quadrilateral (Fig. 44) is a representation of the z-plane by means of $\eta\begin{bmatrix} a, & b, & c & \\ \lambda_0, & \mu_0, & \nu_0 & x \end{bmatrix}$.

47. We discuss next in an analogous manner the case $\lambda_0 + \mu_0 + \nu_0 = 3$. Proceeding in a manner similar to that in Art. 46, we have, corresponding to Fig. 43, instead of Fig. 44, Fig. 46, a quadrilateral which passes once through infinity; that is to say, extends over the whole of the infinite portion of the plane.

In view of the discussion at the end of Art. 44 it is an easy matter, when λ, μ, ν are complex and $\lambda_0 + \mu_0 + \nu_0 =$ either 1 or 3, to ascend from the reduced polygons, as represented in Figs. 44 and 46, to the general polygons by means of lateral and polar attachment of whole-planes. In Figs. 44 and 46, if $\xi(\lambda) \leqq \xi(\mu) + \xi(\nu)$, there must be a lateral attachment of a, b, c whole-planes; if, on the other hand $\xi(\lambda) \geqq \xi(\mu) + \xi(\nu)$, there must be a lateral attachment of b and c whole-planes and a polar attachment of A whole-planes. That is in Fig. 44, the point at infinity will be covered by the sheets of the polygon $a + b + c$ and $A + b + c$ times, respectively. In Fig. 46, the corresponding numbers will be $a + b + c + 1$ and $A + b + c + 1$. These four numbers are in each case equal to $\frac{1}{2}(\lambda + \mu + \nu - 1) = k$, the total number of roots of the equation $\varphi(x_1, x_2) = 0$.

The discussion of special cases; i. e. when the real parts of λ, μ, ν are integral, etc., will be deferred until cases β) and γ) have been discussed.

$$\textit{Case } \beta): \quad -\lambda + \mu + \nu = 2k + 1 .$$

48. The character of the polygons belonging to this case is defined by the formula $-\lambda + \mu + \nu = 2k + 1$, where λ, μ, ν are complex numbers whose real parts are rational, not integral, and k is a positive integer. The discussion of the cases in which the real parts of λ, μ, ν are integral will be given later, in Art. 50.

We begin, as in case α), by placing

$$\lambda = \lambda' + i\lambda'', \quad \mu = \mu' + i\mu'', \quad \nu = \nu' + i\nu''.$$

Since

$$-\lambda + \mu + \nu = 2k + 1,$$

we have

$$-\lambda' + \mu' + \nu' = 2k + 1 \quad \text{and} \quad -\lambda'' + \mu'' + \nu'' = 0 .$$

We now separate the real rational numbers λ', μ', ν' into their integral and fractional parts by the equations,

$$\lambda' = \xi(\lambda') + (\lambda'), \quad \mu' = \xi(\mu') + (\mu'), \quad \nu' = \xi(\nu') + (\nu') .$$

Since $-\lambda' + \mu' + \nu' = 2k + 1$, we shall have two cases to consider,

$$-(\lambda') + (\mu') + (\nu') = 1, \tag{b$_1$}$$

$$-(\lambda') + (\mu') + (\nu') = 0 . \tag{b$_2$}$$

(b_1). $-(\lambda') + (\mu') + (\nu') = 1$. In the presentation of this case we put, for the reduced polygon,

$$\lambda'_0 = (\lambda'), \qquad \mu'_0 = (\mu'), \qquad \nu'_0 = (\nu');$$

$$\lambda_0 = \lambda'_0 + i\lambda'', \quad \mu_0 = \mu'_0 + i\mu'', \quad \nu_0 = \nu'_0 + i\nu''.$$

This is admissible, since the two systems of equations just given satisfy the conditions of our problem; i. e.

$$-\lambda_0 + \mu_0 + \nu_0 = -\lambda'_0 + \mu'_0 + \nu'_0 + i(-\lambda'' + \mu'' + \nu'') = 1 .$$

Let the z-plane be cut from c through b to a. The integral η in (47), β) shows that the vertex of the η-polygon corresponding to the point $z = a$ lies at infinity.

Hence, in analogy with the method employed in the preceding case, the figure which arises by the corresponding loxodromic substitutions (see Fig. 47;

Fig. 41.

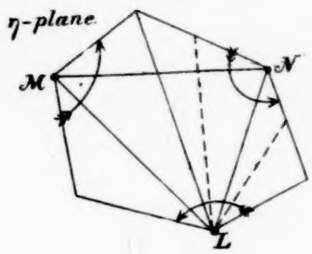


Fig. 42.

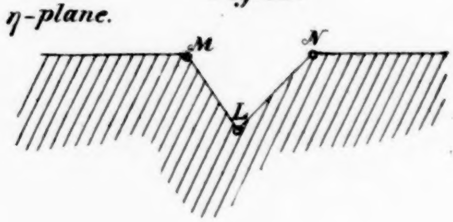


Fig. 43.

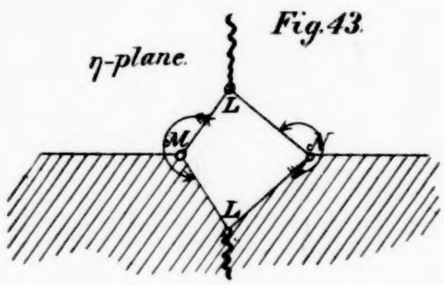


Fig. 44.

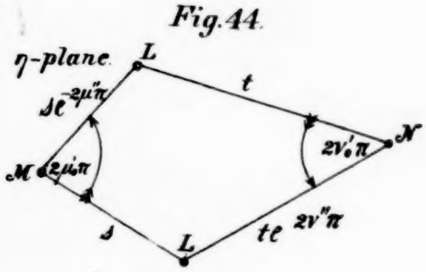


Fig. 46.

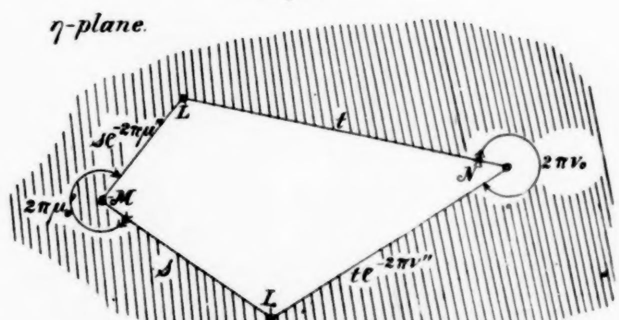


Fig. 45.

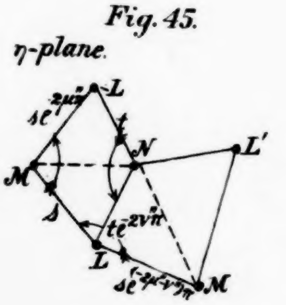


Fig. 47.

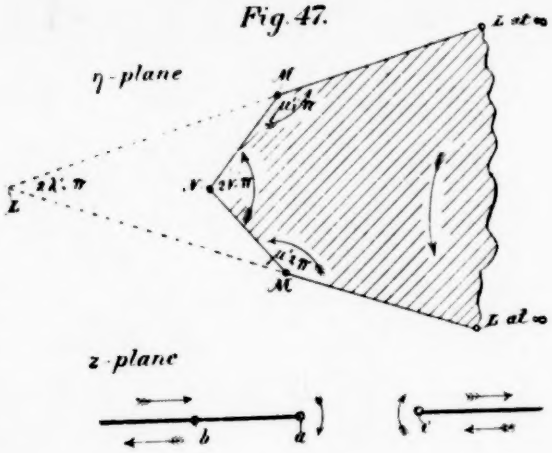


Fig. 48.

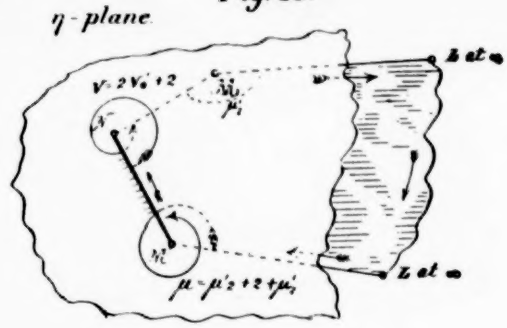


Fig. 49.

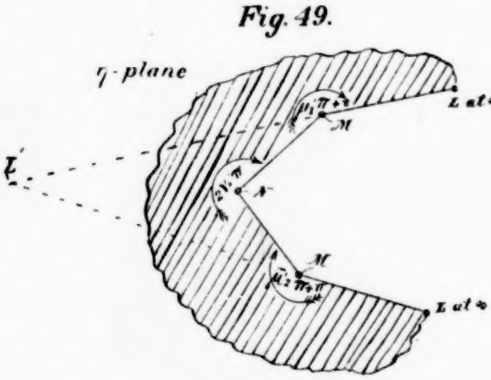


Fig. 50.

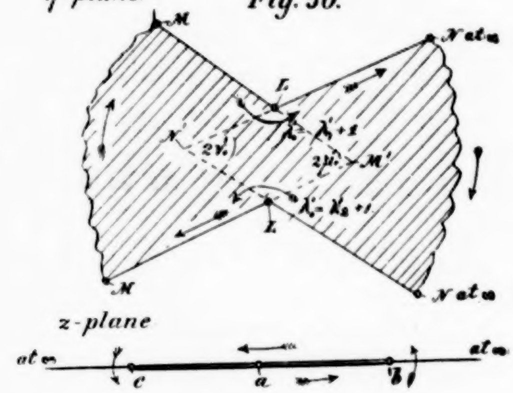


Fig. 51.

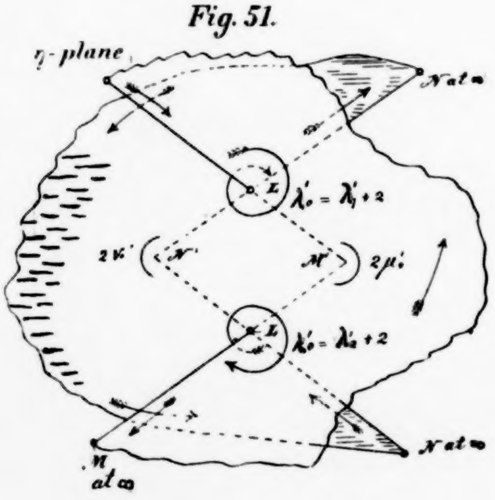


Fig. 52.

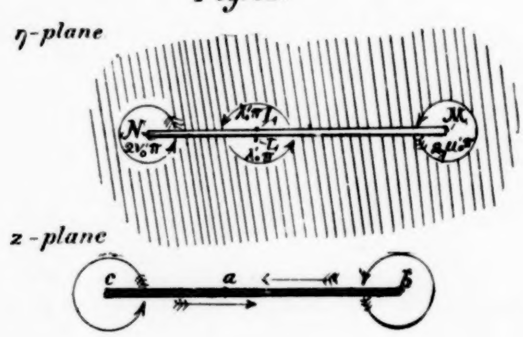
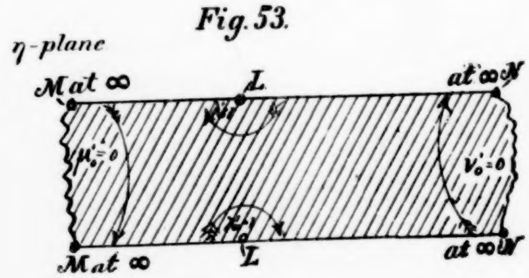


Fig. 53.



also, compare Art. 45) has angles at N and L or L' equal to $2\nu'_0\pi$ and $2\lambda'_0\pi$, respectively, by construction, and the sum of the remaining angles (at M and M') may be demonstrated, as above, by Fig. 45, to be $2\pi\mu'_0$.

That Fig. 47 satisfies the conditions of the problem may be easily seen: If we assume $\mu'_1 + \mu'_2 = 2\mu'_0$, then, from the polygon $MLMNM$, we have

$$2\pi = 2\lambda'_0\pi + \pi - \mu'_2\pi + 2\pi - 2\nu'_0\pi + \pi - \mu'_1\pi ;$$

$$\text{i. e. } 2 = 2\lambda'_0 - 2\mu'_0 - 2\nu'_0 + 4 , \text{ or } -\lambda'_0 + \mu'_0 + \nu'_0 = 1 .$$

We can ascend to the general η-polygon, by means of the system of equations

$$\lambda = \lambda_0 + b + c , \quad \mu = \mu_0 + c + a , \quad \nu = \nu_0 + a + b ,$$

a, b, c being positive integers (see Art. 25). In accordance with the equations just given, the general polygon in Fig. 47 is constructed by the lateral attachment of a, b, c whole-planes to the sides of the reduced η-polygon, as branch-sections.

Since the point at infinity in this case (Fig. 47) is itself a branch point, this point will not be covered by the whole-planes attached to the sides ML above and ML below, but will be covered by the η-polygon a times only.

Fig. 48 illustrates the construction of the η-polygon when

$$\lambda = \lambda_0 , \quad \mu = \mu_0 + 2 , \quad \nu = \nu_0 + 2 .$$

(b_2). Here $(\lambda') = (\mu') + (\nu')$, neither (λ'), (μ'), (ν') being zero. The reduced polygon in this case is characterized by the equations

$$\lambda'_0 = (\lambda') , \qquad \mu'_0 = (\mu') + 1 , \quad \nu'_0 = (\nu') ;$$

$$\lambda_0 = \lambda'_0 + i\lambda'' , \quad \mu_0 = \mu'_0 + i\mu'' , \quad \nu_0 = \nu'_0 + i\nu'' .$$

Proceeding in a manner analogous to that adopted in the derivation of Fig. 47, Fig. 49 is obtained for the reduced polygon in this case. That this figure satisfies the conditions of our problem is shown as follows :—

Assuming

$$2\mu'_0 = 2(\mu') + 2 = \mu'_1 + \mu'_2 + 2 ,$$

the polygon $NMLM$ gives

$$2\pi = (2 - 2\lambda'_0)\pi + \mu_2\pi + \mu_1\pi + 2\nu'_0\pi ;$$

whence

$$2 = 2 - 2\lambda'_0 + 2\mu'_0 - 2 + 2\nu'_0 ,$$

or

$$-\lambda'_0 + \mu'_0 + \nu'_0 = 1 .$$

We may now construct the general γ-polygon by means of the systems of equations

$$\lambda = \lambda_0 + b + c\,, \quad \mu = \mu_0 + c + a\,, \quad \nu = \nu_0 + a + b\,;$$

i. e. by the lateral attachment of a, b, c whole-planes to the four sides of the reduced polygon, Fig. 49.

In Fig. 49, the point L is a branch point; whence the planes attached laterally to the sides ML above and ML below do not cover the point at infinity.

The point at infinity is covered a times only; for, in this case,

$$\tfrac{1}{2}(-\lambda + \mu + \nu - 1) = \tfrac{1}{2}(-\lambda_0 + \mu_0 + \nu_0 + 2a - 1) = a = k\,,$$

the number of roots of $\varphi_k = 0$.

$$\textit{Case } \gamma):\ \lambda - \mu - \nu = 2k + 1\,.$$

49. In this case k is a positive integer and λ, μ, ν, are complex; whence $\lambda > \mu + \nu$.

Retaining the same method of division of λ, μ, ν into real and imaginary parts which was employed in α) and β), we must have, owing to the equation $\lambda - \mu - \nu = 2k + 1$, $\lambda' - \mu' - \nu' = 2k + 1$ and $\lambda'' - \mu'' - \nu'' = 0$.

Again, separating λ', μ', ν' into their integral and fractional parts, we have the cases to consider:

$$(\lambda') - (\mu') + (\nu') = 0\,, \quad \text{but} \quad (\lambda'), (\mu'), (\nu') \text{ not} = 0\,; \qquad (c_1)$$

$$(\lambda') - (\mu') + (\nu') = -1\,. \qquad (c_2)$$

(c_1). $(\lambda') - (\mu') + (\nu') = 0$, but (λ'), (μ'), (ν') not $= 0$. In this case the reduced polygon may be defined by the equations

$$\lambda'_0 = (\lambda') + 1\,, \quad \mu'_0 = (\mu')\,, \quad \nu'_0 = (\nu')\,;$$

$$\lambda_0 = \lambda'_0 + i\lambda''\,, \quad \mu_0 = \mu'_0 + i\mu''\,, \quad \nu_0 = \nu'_0 + i\nu''\,.$$

In a manner analogous to that employed in cases α) and β), just preceding, we obtain for our reduced γ-polygon, Fig. 50. That Fig. 50 satisfies the conditions of the problem may be easily seen as follows:—

Assuming

$$\lambda'_1 + \lambda''_2 + 2 = 2\lambda'_0 = 2(\lambda') + 2\,,$$

we have, from polygon $LN'LM'$, Fig. 50,

$$2\nu'_0 + 2\mu'_0 + 2 - \lambda'_1 + 2 - \lambda'_2 - 2 = 2\,;$$

$$\text{i. e.} \quad \lambda'_0 - \mu'_0 + \nu'_0 = 0\,.$$

Since $\lambda > \mu + \nu$, we may, as in the corresponding case of Art. 26, ascend to the general η-polygon by means of the equations

$$\lambda = \lambda_0 + b + c + 2A\,, \quad \mu = \mu_0 + c\,, \quad \nu = \nu_0 + b\,.$$

This general polygon is constructed by the lateral attachment of $b + c$ whole-planes and the polar attachment of A whole-planes to the sides of the reduced polygon in Fig. 50. The general surfaces will cover the point at infinity A times only, since M and N are branch points at infinity.

(c_2). $(\lambda') - (\mu') - (\nu') = -1$. We take for our reduced polygon

$$\lambda'_0 = (\lambda') + 2\,, \quad \mu'_0 = (\mu')\,, \qquad \nu'_0 = (\nu')\,;$$

$$\lambda_0 = \lambda'_0 + i\lambda''\,, \quad \mu_0 = \mu'_0 + i\mu''\,, \quad \nu_0 = \nu'_0 + i\nu''\,;$$

$$2\lambda'_0 = 2(\lambda') + 4 = \lambda'_1 + \lambda'_2 + 4\,.$$

In the same manner as the reduced polygons were derived in the preceding cases, we may show that in this case our reduced polygon will have the form given in Fig. 51. That Fig. 51 satisfies the conditions of the problem is shown as follows :—

From the polygon $LN'LM'$ we have

$$2 = 2 - 2\nu'_0 + 2 - 2\mu'_0 + \lambda'_1 + \lambda'_2\,;$$

i. e. $$2 = 2 - 2\nu'_0 + 2 - 2\mu'_0 + 2\lambda'_0 - 4\,,$$

or $$\lambda'_0 - \mu'_0 - \nu'_0 = 1\,.$$

We may construct the η-polygon for the values $\lambda = \lambda_0 + 4$, $\mu = \mu_0$, $\nu = \nu_0$, by the polar attachment, in Fig. 51, of a whole-plane along a branch-line drawn from L above to L below. The general polygon may be constructed by means of the equations

$$\lambda = \lambda_0 + b + c + 2A\,, \quad \mu = \mu_0 + c\,, \quad \nu = \nu_0 + b\,;$$

i. e. by the lateral attachment of b and c whole-planes to the sides of the polygon in Fig. 51 and the polar attachment of A whole-planes to a branch-section running from L above to L below.

The point at infinity will be covered A times.

50. λ, μ, ν *complex; but their real parts all integral, one integral, or one zero.*

In what precedes we have shown how the six reduced polygons, α) Figs. 40, 46; β) Figs. 47, 49; γ) Figs. 50, 51 were derived, assuming λ, μ, ν to be complex. In Art. 17 it was shown that equation (45) holds for all values of

λ, μ, ν, whether λ, μ, ν, being real are integral, or being complex their real parts are integral. If the real parts of λ, μ, ν are all integral, the reduced polygons corresponding to Figs. 40 and 47 vanish, while those corresponding to Figs. 46 and 49 reduce to Fig. 52, in which, however, one of the points, say L, may lie at infinity. In the case illustrated by Fig. 46, retaining the notation used in the discussion of cases α), β), γ), (λ, μ, ν complex), if λ', μ', ν' are integers, we place for our reduced polygon,

$$\lambda'_0 = 1\,, \qquad \mu'_0 = 1\,, \qquad \nu'_0 = 1\,,$$

$$\lambda_0 = \lambda'_0 + i\lambda''\,, \quad \mu_0 = \mu'_0 + i\mu''\,, \quad \nu_0 = \nu'_0 + i\nu''\,.$$

The form of the reduced polygon is shown in Fig. 52.

In the case illustrated by Fig. 47, the form and position of the reduced triangle is the same, except that the vertex L lies at infinity.

If, in the cases illustrated by Figs. 50 and 51, λ', μ', ν' are positive integers the corresponding reduced polygons become that shown in Fig. 53. The arithmetical reduction belonging to this case will be

$$\lambda'_0 = 1\,, \quad \mu'_0 = 0\,, \quad \nu'_0 = 0\,;$$

whence

$$2\lambda'_0 - 2\mu'_0 - 2\nu'_0 = 2\,,$$

or

$$\lambda'_0 - \mu'_0 - \nu'_0 = 1\,,$$

as required.

It is a simple matter after the discussion for complex λ, μ, ν (the real parts of λ, μ, ν not being integral) to show how to form the general polygons from the reduced-polygons shown in Figs. 52 and 53.

If a single λ', μ', ν' is integral, we may construct the reduced-polygons corresponding to the reduced-polygons, α) Figs. 40, 47 ; β) Figs. 46, 49 ; γ) Figs. 50, 51 by modifying them in the proper manner ; and so, likewise, if either λ', μ', ν' are zero.

IV.

51. In this section we propose to show what triangles in II define the algebraic integrals of the differential equation,

$$\frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \cdot \frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)} \cdot y = 0\,,$$

found by Schwarz in Crelle's Journal, Vol. 75, § 1, from 1^a to 4^c.

We shall retain Schwarz's notation throughout this discussion, putting

$$\lambda^2 = (1-\gamma)^2\,, \quad \mu^2 = (\alpha - \beta)^2\,, \quad \nu^2 = (\gamma - \alpha - \beta)^2\,,^*$$

and consider simply the cases in which λ, μ, ν are positive.

* Schwarz's paper, § 3.

52. We shall not take space to repeat the various conditions under which the several integrals mentioned by Schwarz are algebraic integrals.

In 1^a of Schwarz's paper, $F(a, \beta, \gamma, x) = F(-n, \beta, \gamma, x)$, under the conditions named by him, is an integral algebraic integral when $a = -n$. From Art. 51 we have

$$\lambda = 1 - \gamma, \quad \mu = a - \beta, \quad \nu = \gamma - a - \beta;$$

whence $\lambda - \mu + \nu$ is a positive odd integer $= 2n + 1$. By interchanging λ and μ, this falls under case β) in Art. 33.

53. In 1^b, under the conditions named,

$$x^{1-\gamma} F(a - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x) = x^{n'} F(-n + n', \beta + n', 1 + n, x)$$

is an integral algebraic function when $1 - \gamma = n'$ and $a - \gamma + 1 = -n + n'$, n and n' being integers. In our notation λ will be integral and $\lambda - \mu + \nu = 2n + 1$, hence this case falls under β), in Art. 35, (b_4).

54. 1^c is embraced by Schwarz in the discussion of 1^a and 1^b.

55. 2^a furnishes the conditions that the integral

$$(1 - x)^{\gamma - a - \beta} F(\gamma - a, \gamma - \beta, \gamma, x) = x^c F(n + \gamma + c, -n, \gamma, x)$$

be an algebraic function; namely, $\gamma - \beta = -n$, a negative integer (c being a rational number). In our notation, from Art. 51 we have $\lambda - \mu - \nu = 2n + 1$, a positive odd integer. The case 2^a falls under II, γ).

56. In 2^b

$$\lambda = 1 - \gamma = n', \quad \beta = 1 + n - n',$$

n and n' being positive integers. λ is thus a positive integer; μ, positive and rational; ν, negative and rational. Hence, interchanging λ and ν, 2^b falls under II, β), (b_4).

57. The discussion of 2^c is embraced under that of 2^a and 2^b.

58. In 3^a $a - \gamma + 1 = -n$; whence, since

$$\lambda = 1 - \gamma, \quad \mu = a - \beta, \quad \nu = \gamma - a - \beta,$$

we have

$$-\lambda - \mu + \nu = 2n + 1,$$

a positive odd integer. By interchanging λ and ν, this case falls under II, γ).

59. In 3^b

$$a = -n + n', \quad \beta = n' - n'', \quad \gamma = 1 + n',$$

n, n', n'' being positive integers; whence

$\lambda = 1 - \gamma = -n'$, a negative integer,

$\mu = a - \beta = -n + n''$, a negative integer,

$\nu = \gamma - a - \beta = 1 + n - n' + n''$, a positive integer,

since

$$1 \leqq n' < n\,, \quad 0 \leqq n'' < n'\,.$$

That is, λ, μ, ν are all integral, λ and μ being negative ; hence by interchanging λ and ν, 3^b falls under the case of integral values of λ, μ, ν in II, γ), (c_2).

60. Case 3^c is embraced under those considered in Arts. 58 and 59.

61. 4^a has $\beta = 1 + n$, an integral number; hence, taking the negative values of λ, μ, ν, in 51, we obtain $\lambda + \mu + \nu = 2n + 1$, an odd positive integer. This case falls under II, a).

62. From 4^b

$$\gamma - a = -n'' + n'\,, \quad \gamma - \beta = -n + n'\,, \quad \gamma = 1 + n'\,;$$

$$\gamma - a - \beta = c = -n'' + n' - n - 1\,;$$

n, n', n'', being integers. Hence, taking the negative values of λ, μ, ν from Art. 51, we have $\lambda = n'$, a positive integer; $\mu = n - n''$, a positive integer; $\nu = -\gamma + a + \beta = n'' - n' + n + 1$, a positive integer. 4^b is, therefore, embraced in the integral case discussed under II, a).

63. 4^c is embraced in the discussion of 4^a and 4^b.

April 1, 1892.

CONTENTS.

Page.

On certain Determinant Forms and their Applications. (Second paper—*Continued*.) By W. H. Echols, ... 125

Forsyth's Theory of Functions of a Complex Variable. By W. H. Echols, ... 143

A Study of Certain Special Cases of the Hypergeometric Differential Equation. By J. H. Boyd, ... 145

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